Tempered fundamental groups and Semi-graphs of anabelioids

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Motivation

- Setting: non-archimedean local case.
- Goal: recover information of infinite topological (combinatorial/analytic) covers.
- Example of interesting infinite cover: the uniformization of Tate curves by $\mathbb{G}_m$ (cf. the Etale Theta function).
- $\rightsquigarrow$ Tempered fundamental group: a (not necessarily profinite) topological group.
Outline

1. Tempered fundamental groups
   - Tempered fundamental groups of curves
   - Temperoids
   - Semigraphs of anabelioids

2. Tempered recovering of graphs
   - Abstract case
   - Case of a geometric curve
   - Semi-graphs with arithmetic action

3. Other results
   - Tempered anabelian geometry
   - Decomposition and inertia subgroups
   - Decomposition groups of subgraphs
1. **Tempered fundamental groups**
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X/K be a connected smooth curve over a nonarchimedean field.
(Si)i∈I: cofinal system of finite étale Galois covers of X.
S∞i: universal topological cover of analytic sp. Xan.

1 → π1top(Si) → Gal(S∞i/X) → Gal(Si/X) → 1

Definition

Btemp(X): cat. of countable coproducts of quotients of some S∞i.

π1temp(X) := lim ← i i Gal(S∞i/Xan).

(well defined up to inner isomorphism)

Pointed definition:

¯x: geometric point of Xan.

F¯x : S → S¯x.

Then π1temp(X, ¯x) = Aut(F¯x).
Profinite completion: $\hat{\pi}_1^{\text{temp}}(X) = \hat{\pi}_1(X)$ is the profinite étale fundamental group.

$\overline{K}$: completion of an algebraic closure of $K$.

$$1 \to \pi_1^{\text{temp}}(X_{\overline{K}}) \to \pi_1^{\text{temp}}(X) \to G_K \to 1$$

$B^{\text{temp}}(X) \to \pi_1^{\text{temp}}(X)$—CountSet is an equivalence of categories.
Example: $K = \mathbb{C}_p$, $X$: elliptic curve.

\[ \ldots \rightarrow X \rightarrow \ldots \rightarrow X \rightarrow X \]

- **Good reduction**: $X^\infty = X^\text{an}$, $\pi_1^\text{temp}(X) = \widehat{\pi}_1(X)$.
- **Bad reduction (Tate curve)**: $X^\text{an} \cong \mathbb{G}_m^\text{an} / q^\mathbb{Z}$, $X^\infty \cong \mathbb{G}_m^\text{an}$

\[ \ldots \rightarrow \mathbb{G}_m^\text{an} \rightarrow \ldots \rightarrow \mathbb{G}_m^\text{an} \rightarrow \mathbb{G}_m^\text{an} \]

\[ \ldots \rightarrow X^\text{an} \rightarrow \ldots \rightarrow X^\text{an} \rightarrow X^\text{an} \]

\[ 1 \rightarrow \widehat{\pi}_1(\mathbb{G}_m) \rightarrow \pi_1^\text{temp}(X) \rightarrow \pi_1^\text{top}(X^\text{an}) \rightarrow 1 \]

\[ \parallel \quad \parallel \quad \parallel \]

\[ 1 \rightarrow \widehat{\mathbb{Z}}(1) \rightarrow \pi_1^\text{temp}(X) \rightarrow \mathbb{Z} \rightarrow 1 \]
A topological group which an inverse limit of an inverse system of surjections of countable discrete topological groups and whose topology has a countable basis is a \textit{tempered} group.

If \( \Pi \) is a tempered group, the category of countable discrete \( \Pi \)-sets is denoted by \( B^{\text{temp}}(\Pi) \). Such a category is called a \textit{temperoid}.

A morphism of temperoids \( \phi : \mathcal{T}_1 \rightarrow \mathcal{T}_2 \) is a functor \( \mathcal{T}_2 \rightarrow \mathcal{T}_1 \) which preserves finite limits and countable colimits.

Example: if \( K \subset \mathbb{C}_p \).

- \( \pi^{\text{temp}}_1(X) \) is a tempered group.
- \( B^{\text{temp}}(X) \) is a temperoid.
If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two temperoids, let $\text{Hom}(\mathcal{T}_1, \mathcal{T}_2)$ be the category of morphisms from $\mathcal{T}_1$ to $\mathcal{T}_2$.

**Proposition**

Let $\Pi_1, \Pi_2$ be two tempered groups and let $\mathcal{T}_1, \mathcal{T}_2$ be the corresponding temperoids. The category $\text{Hom}(\mathcal{T}_1, \mathcal{T}_2)$ is naturally equivalent to the category whose:

- **object** are continuous group homomorphisms $\Pi_1 \to \Pi_2$.
- **morphisms** $\phi \to \psi$ are elements $g \in \Pi_2$ such that $\gamma_g \circ \phi = \psi$. 
Semigraphs of anabelioids: motivating examples

- $X_K$: hyperbolic (log) curve over $K$ finite ext. of $\mathbb{Q}_p$;
- $\overline{K}$: (completion of) an algebraic closure of $K$;
- $X_K^{\log}$: log stable model of $X_K$; $X_S^{\log}$: special fiber;
- $G_c$: semi-graph of anabelioids of $X_S^{\log}$:
  - $V(G_c)$: irred. components of $X_S^{\log}$;
  - $E(G_c)$: nodes and cusps of $X_S^{\log}$;
  - $G_v^c$: cat. of log-étale covers of the irred. comp. $U_v$ corr. to $v$, $\simeq$ cat. of tame covers of $U_v^{tr}$;
  - $G^c_\varnothing$: cat. of log-étale covers of $e$, $\simeq B(\mathbb{Z}(\neq p))$.
- $\hat{\pi}_1(X_{\overline{K}}) \rightarrow \hat{\pi}_1(G_c)$;
- $\hat{\pi}_1(X_{\overline{K}})^\Sigma \simeq \hat{\pi}_1(G_c)^\Sigma$ if $p \not\in \Sigma$. 
Covers of semigraphs

Let $\mathcal{G} = (G, (G_v)_{v \in V(G)}, (G_e)_{e \in E(G)}, (b^*: G_v \to G_e)_{b:v \to e})$ be a countable semigraph of anabelioids. Here $b^*: G_v \to G_e$ is an exact functor (corresponding to $b_*: \pi_1(G_e) \to \pi_1(G_v)$).

Definition

A cover $S$ of $\mathcal{G}$ is given by a collection (of 1-dimensional descent data): $S = ((S_v)_{v \in V(G)}, (\phi_e)_{e \in E(G)})$, where

- $S_v$ is an object of $G_v^\top$;
- $\phi_e$ is an isomorphism $b_1^*S_{v_1} \simeq b_2^*S_{v_2}$, if $b_1: e \to v_1, b_2: e \to v_2$ are the two branches of $e$.

This gives a category $\mathcal{B}^{\text{cov}}(\mathcal{G})$. 
Temperer covers

Definition

- $S$ is *finite* if $S_v$ is finite for every vertex $v$.
- A morphism $f = (f_v)_v : S \to S'$ of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ is *topological* if for all $v \in V(\mathcal{G})$ and every connected component $s$ of $S_v$, $(f_v)|_s : s \to f(s)$ is an isomorphism.
- A cover $S$ of $\mathcal{G}$ is *tempered* if there exists a finite cover $T$ of $\mathcal{G}$ such that $S \times T \to T$ is a topological cover.

$\mathcal{B}^{\text{temp}}(\mathcal{G})$ is the full subcategory of $\mathcal{B}^{\text{cov}}(\mathcal{G})$ given by countable coproducts of tempered covers.
$S \rightsquigarrow$ s-gph of anab. $S$ over $\mathcal{G}$:

- $S_v = \coprod_v \pi_0(S_v)$;
- $S_e = \coprod_e \pi_0(S_e)$;
- if $c' \in \pi_0(S_c)$, $S_{c'} = (G_v)_c (\simeq B(\text{Stab}_{\pi_1(G_v)}(\bar{c}'))$ for any $\bar{c}' \in c'$).

$\mathcal{B}^\text{cov}(S) \simeq B^\text{cov}(\mathcal{G})_S$.

The full subcategory $\mathcal{B}^\text{top}(S)$ of $\mathcal{B}^\text{cov}(\mathcal{G})_S$ defined by topological morphisms $S' \to S$ is equivalent to the category of covers of $S$ with countable fiber.

In particular, if $S$ is connected one gets a universal topological cover $S^\infty \to S$. 

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Tempered fundamental group.

\((S_i)_{i \in I}\): cofinal collection of finite Galois covers of \(\mathcal{G}\);
\(S_i^\infty\): universal topological cover of \(S_i\).

**Definition**

The tempered fundamental group of \(\mathcal{G}\) is the group:

\[
\pi_1^{\text{temp}}(\mathcal{G}) = \lim_{i \leftarrow} \text{Gal}(S_i^\infty/\mathcal{G}).
\]

well defined up to inner isomorphism.

\(c\): edge or vertex of \(\mathcal{G}\); \(F : \mathcal{B}(\mathcal{G}_c) \to \text{fSet}\): fiber functor;
\(F' : \mathcal{B}^{\text{temp}}(\mathcal{G}) \to \text{CountSet}\) defined by \(S \mapsto F^T(S_v)\);
then \(\pi_1^{\text{temp}}(\mathcal{G}) = \text{Aut}(F')\).
Technical conditions:

- q-coherent: approximation by semi-graph of (uniformly) finite anabelioids.
- coherent: q-coh. + components top. finitely generated.
- totally elevated: for every \( v \), one can find "big" sgps in \( G'_v \) not trivially intersecting all the edge components for some approximator \( G' \).
- totally estranged: for every \( e, e' \sim v \) and \( g \in \Pi_v \), \( g\Pi_eg^{-1} \cap \Pi_{e'} \neq 1 \) only if \( e = e' \) and \( g \in \Pi_e \).
- vertically slim: vertical components are slim.
Proposition (prop. 3.6)

- $\pi_1^\text{temp}(\mathcal{G})$ is tempered, residually finite, and temp-slim.
- There is a natural equivalence of categories $\mathcal{B}^\text{temp}(\pi_1^\text{temp}(\mathcal{G})) \rightarrow \mathcal{B}^\text{temp}(\mathcal{G})$.
- The profinite completion of $\pi_1^\text{temp}(\mathcal{G})$ is $\hat{\pi}_1(\mathcal{G})$.
- The tempered fundamental group is functorial.
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\(_c : B^\text{temp}(\mathcal{G}) \to \mathcal{G}_c^\top\) induces an injective morphism \(\hat{\pi}_1(\mathcal{G}_c) \to \pi^\text{temp}_1(\mathcal{G}),\) defined up to inner automorphism. Its image is called a \textit{vertical/edge-like} subgroup.

\[S^\infty := \lim_{\leftarrow i} S_i^\infty \text{ (pro-graph + action of } \pi^\text{temp}_1(\mathcal{G})).\]

If \(\nu\) pro-vertex of \(S^\infty\), then \(D_\nu = \text{Stab}_{\pi^\text{temp}_1(\mathcal{G})}(\nu)\) is a vertical group.

**Theorem (Maximal compact subgroups; th. 3.7)**

- The maximal compact subgroups of \(\pi^\text{temp}_1(\mathcal{G})\) are the vertical subgroups.
- The nontrivial intersections of two distincts maximal compact subgroups are the edge-like groups.
• Action of a finite group (without flipped edge) on a tree has a fixed vertex.

• Apply this to the action of a compact subgroup on $S_i^\infty$.

• Non-vanishing of the projective system of fixed vertices: take the profinite completion and use that, cofinally, at most two adjacent vertices are fixed (estrangedness condition).
$f : \Pi_1 \to \Pi_2$ is \textit{q-geometric} if $f(\max \text{ cpct sgp}) \subset_{\text{op}} \max \text{ cpct sgp}$

(+ similar condition for nontrivial intersections of max cpct sgp)

\begin{corollary}
Let $\mathcal{G}$ and $\mathcal{H}$ be graphs of anabelioids. Any q-geometric morphism $\pi_1^{\text{temp}}(\mathcal{G}) \to \pi_1^{\text{temp}}(\mathcal{H})$ is induced by a locally open morphism $\mathcal{G} \to \mathcal{H}$.
\end{corollary}
$K/\mathbb{Q}_p$: finite extension; $\overline{K}$: algebraic closure of $K$;
$G_K = \text{Gal}(\overline{K}/K)$;
$X_K$: hyperbolic (log) curve;
$\Pi = \pi_1^{\text{temp}}(X)$, $\Delta = \pi_1^{\text{temp}}(X_{\overline{K}})$.

$$1 \to \Delta \to \Pi \to G_K \to 1$$

$X_{\log \mathcal{O}_K}^\log$: log stable model; $X_s^\log$: special fiber;
$\mathcal{G}^c$: semi-graph of anabelioids of $X_s^\log$;
$\mathcal{G}$: graph of anab. obtained by forgetting cusps.
Natural quotient map $\Delta \to \pi_1^{\text{temp}}(\mathcal{G})$.

If $p \notin \Sigma$, $\Delta^\Sigma = \pi_1^{\text{temp}}(\mathcal{G}^\Sigma)$,
where $\Delta^\Sigma = \varprojlim_H H$, $H$ goes through ($\Sigma$-finite)-by-($\text{torsion-free}$) quotients of $\Delta$. 
Theorem (Cor. 3.11)

Let $X_\alpha/K_\alpha$, $X_\beta/K_\beta$ be log smooth curves. Every isomorphism $\Delta_\alpha \simeq \Delta_\beta$ induces an isomorphism $G^c_\alpha \simeq G^c_\beta$.

Idea:

- Apply $(\Sigma$-finite)-by-(torsion-free) completion and reconstruction theorem to get an isomorphism $G^\Sigma_\alpha \simeq G^\Sigma_\beta$.
- Apply the same thing to corr. finite-open sgps $\Delta'_{\square}$ of $\Delta_{\square}$.
- Cusp-sgps and edge-like sgps in $\Delta^\Sigma_{\square}$ are the max. sgps commensurable to images of edge-like subgroups of some $(\Delta'_{\square})^\Sigma$. 
A finite-open normal sgp $\Delta' \subset \Delta$ comes from a finite cover of $G^c$ iff:

- $(\Delta')^\Sigma \to \Delta^\Sigma$ is quasi-geometric;
- $\text{Stab}_{\Delta/\Delta'}(e)$ prime-to-$p$;
- $\text{Stab}_{\Delta/\Delta'}(v)$ acts faithfully on $(G'_v)^\Sigma$ (the inertia sgp at $v$ is trivial).

(the inertia in $(G'_v)^\Sigma$ is pro-$p \leadsto$ recovers $p$)

$\pi^\text{temp}_1(G)$: proj.lim. of discrete quotients of $\Delta$ given by extension of such quotients by a torsion-free group.
Let $\mathcal{G}$ be a semigraph of anabelioids (+ technical conditions).
Let $\mathcal{A}$ be a pointed slim connected anabelioid.
Let $\rho : \hat{\pi}_1(\mathcal{A}) \to \mathcal{G}$ such that:

- $\hat{\pi}_1(\mathcal{A})$ is top. fin. generated;
- $\mathcal{G}$ is locally finite;
- $\exists H \subset_{op} \hat{\pi}_1(\mathcal{A})$, s.t.:
  - $H$ acts trivially on $\mathcal{G}$;
  - $H \to \text{Out}(\hat{\pi}_1(\mathcal{G}_v))$ is continuous;
  - $\exists$ finite $V$, s.t. for every $w$, there is $v$ and $H$-compatible $\mathcal{G}[v] \sim \mathcal{G}[w]$.
  (here so that $\rho$ partially extends to tempered covers)
Let $\mathcal{G} = (\mathcal{G}, \mathcal{A}, \rho_{\mathcal{G}})$ be an arithm. s-gph. of anab.
There is a natural extension of $\hat{\pi}_1(\mathcal{A})$ by $\pi^\text{temp}_1(\mathcal{G})$:

$$1 \to \pi^\text{temp}_1(\mathcal{G}) \to \pi^\text{temp}_1(\mathcal{G}) \to \hat{\pi}_1(\mathcal{A}) \to 1.$$ 

Alternative notations: $1 \to \Pi^\text{temp}_\mathcal{G} \to \Pi^\text{temp}_\mathcal{G} \to \Pi_\mathcal{A} \to 1$.

Connected $\pi^\text{temp}_1(\mathcal{G})$-sets come from $\mathcal{G}' = (\mathcal{G}', \mathcal{A}', \rho_{\mathcal{G}'})$ above $\mathcal{G}$ s.t.:

- $\hat{\pi}_1(\mathcal{A}') \subset_{op} \hat{\pi}_1(\mathcal{A})$,
- $\mathcal{G}' \to \mathcal{G}$ comes from a connected tempered cover.
vertical and edge-like subgroups: $\Pi_{\Gamma, c}^{\text{temp}} \subset \Pi_{\Gamma}^{\text{temp}}$.

$\Pi_{\Gamma, c}^{\text{temp}}$ commensurator of $\Pi_{\Gamma, c}^{\text{temp}}$ in $\Pi_{\Gamma}^{\text{temp}}$.

$H \subset \Pi_{\Gamma}^{\text{temp}}$ is arithm. ample if its image in $\Pi_{A}^{\text{temp}}$ is open.

Assume $G, H$ are arithm. estranged: if $b, b' \sim v$, $g \in \Pi_{\Gamma, v}^{\text{temp}}$,

$\Pi_{\Gamma, b}^{\text{temp}} \cap g\Pi_{\Gamma, b'}^{\text{temp}}g^{-1} \subset \Pi_{\Gamma, v}^{\text{temp}}$ is not arithm. ample.

**Theorem (Reconstruction of arithmetic graph of anab., thm 5.4)**

- **Every arithm. ample compact sgps is contained in a vert. sgp and at most two (which are then edge-joined).**

- **There is a bijection between outer arithm. q-geom. morphism $\Pi_{\Gamma}^{\text{temp}} \rightarrow \Pi_{H}^{\text{temp}}$ over $\widetilde{\pi}_1(A)$ and loc. open morphism $\widetilde{G} \rightarrow \widetilde{H}$ over $A$.**
- $K/Q_p$: finite extension.
- $\overline{K}$: algebraic closure of $K$.
- $G_K = \text{Gal}(\overline{K}/K)$.
- $X^\text{log}_K$: smooth log curve.
- $\Pi = \pi_1^\text{temp}(X)$, $\Delta = \pi_1^\text{temp}(X_{\overline{K}})$.

$$1 \rightarrow \Delta \rightarrow \Pi \rightarrow G_K \rightarrow 1$$
Recovering transition maps

$(\Pi_i)_i$: cofinal system of finite-open characteristic sgps of $\Pi$.

$\Pi_i \rightsquigarrow X_i^{\log} \rightarrow K_i$ finite cover of $X_K^{\log}$.

Assume stable reduction $\rightsquigarrow \bar{G}_i$.

$\Pi_i \subset \Pi_j \rightsquigarrow X_i^{\log} \rightarrow X_j^{\log}$

$\rightsquigarrow X_i^{\log}_{\mathcal{O}_K} \rightarrow X_j^{\log}_{\mathcal{O}_K}$ $\rightsquigarrow$ generalized $\bar{G}_i \rightarrow \bar{G}_j$.

$\bar{G}_i \rightarrow \bar{G}_j$ can be recovered from $f_{ij}: \Pi_{\bar{G}_i}^{\text{temp}} \rightarrow \Pi_{\bar{G}_j}^{\text{temp}}$:

- if $f_{ij}(\Pi_{\bar{G}_i,c}^{\text{temp}}) \subset \Pi_{\bar{G},e}^{\text{temp}}$ for a (unique) edge $e$, then $c$ maps to $e$.

- Otherwise $f_{ij}(\Pi_{\bar{G}_i,c}^{\text{temp}}) \subset \Pi_{\bar{G},v}^{\text{temp}}$ for a unique vertex $v$, then $c$ maps to $v$.

(one can also recover the cusps from $\Pi_i \hookrightarrow \Pi_j$).
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Let $K, L$ finite extensions of $\mathbb{Q}_p$
Let $X_K/K$, $Y_L/L$ be two hyperbolic curves

**Theorem (Tempered relative anabelian theorem)**

There is bijection between:

- **dominant morphisms of schemes**;
- **outer morphisms** $\pi_1^{\text{temp}}(X_K) \rightarrow \pi_1^{\text{temp}}(Y_L)$ with image dense in a finite open subgroup such that there is a field morphism $L \rightarrow K$ such that:

\[
\begin{array}{ccc}
\pi_1^{\text{temp}}(X_K) & \rightarrow & \pi_1^{\text{temp}}(Y_K) \\
\downarrow & & \downarrow \\
G_K & \rightarrow & G_L
\end{array}
\]

- $Rk$: the topology of $K$ and $L$ can be recovered from their field structure, so that any field morphism $L \rightarrow K$ is continuous.
- Similarly, every dominant morphism $X_K \rightarrow Y_L$ is over $\mathbb{Q}_p$.
- Idea: take the profinite completion, and apply Grothendieck anabelian conjecture (and use that $N_{\pi_1(X)}(\pi_1^{\text{temp}}(X)) = \pi_1^{\text{temp}}(X)$).
Theorem (tempered vs profinite; thm. 6.6)

Every outer isomorphism $\hat{\pi}_1(X_K) \simeq \hat{\pi}_1(Y_L)$ arises from a unique outer isomorphism $\pi_1^{\text{temp}}(X_K) \simeq \pi_1^{\text{temp}}(Y_L)$.

- A morphism $\hat{\pi}_1(X_K) \simeq \hat{\pi}_1(Y_L)$ is semi-absolute and graphic: $\sim \equiv \overline{G}_{X_K} \simeq \overline{G}_{Y_L}$.
- Apply this to open normal sgps to get topological fundamental groups.
- Use arithm. ampleness to recover transition maps.
Decomposition and inertia groups

- $x$: closed point of $\overline{X}_K$ (compactification of $X_K$);
- $D_x \subset \Pi^{\text{temp}}_{X_K}$: decomposition group;
- if $x$ cusp $I_x = D_x \cap \Delta^{\text{temp}}_{X_K} \cong \hat{\mathbb{Z}}(1)$.

**Theorem (theorem 6.5)**

- $x$ is determined by conj. cl. of $D_x$;
- if $x$ is cusp, $x$ is determined by conj. cl. of $l_x$;
- $D_x$ is commensurably terminal;
- if $x$ is cusp, $C_{\Pi^{\text{temp}}_{X_K}}(l_x) = D_x$;
- Every isomorphism $\pi_1^{\text{temp}}(X_K) \cong \pi_1^{\text{temp}}(Y_L)$ preserves cuspidal decomposition and inertia groups.
Theorem (theorem 6.8)

- If $X_K$, $Y_L$ are once-punctured elliptic curves, every isomorphism $\pi_1^{\text{temp}}(X_K) \simeq \pi_1^{\text{temp}}(Y_L)$ preserves the decomposition groups of torsion points;

- If $X_K$, $Y_L$ are isogenous to genus 0, curves, every isomorphism $\pi_1^{\text{temp}}(X_K) \simeq \pi_1^{\text{temp}}(Y_L)$ preserves the decomposition groups of algebraic points (i.e. defined over a number field).

- Elliptic cuspidalization.

- Belyi cuspidalization.
Proposition (IUTT1, cor. 2.5)

Let \( I \subset \hat{\Delta}_X \) be the decomposition group of a cusp. If \( I \subset \Delta^\text{temp}_X \), then \( I \) is a decomposition group in \( \Delta^\text{temp}_X \).

(+ similar result for decomposition groups of closed points in \( \prod^\text{temp}_X \)).

- Take the image by \( p : \Delta^\text{temp}_X \rightarrow \pi^\text{temp}_1(G_X) \):
  \( \hat{p}(I) = I_c \) for some pro-cuspidal edge of \( \hat{S} = \lim \leftarrow S_i \).
  \( p(I) \) is a nontrivial compact subgroup of \( \pi^\text{temp}_1(G_X) \), so there is \( \nu \in S^\infty \subset \hat{S} \) s.t. \( p(I) \subset D_\nu \).
  \( D_{\nu'} \subset D_\nu \) implies \( c \sim \nu \), therefore \( c \in \hat{S} \).
  Therefore \( I \) is a cuspidal group in \( \Delta^\text{temp}_X \) up to conjugation by an element in Ker \( p \).

- Apply the same to \( J \subset_{f-op} \Delta^\text{temp}_X \).
Decomposition groups of subgraphs

- $\mathbb{H}$: $G_K$-stable connected subgraph of $G_X$;
- $\mathcal{H}$: semi-graph of anabelioids on $\mathbb{H}$ induced by $\mathcal{G}$;
- $\Delta_{\mathcal{H}}^{\text{temp}} \rightarrow \Delta_{\mathcal{G}}^{\text{temp}}$ is injective.
- $\Delta_{\mathcal{H}}^{\text{temp}}$ can be identified with the stabilizer of a pro-connected component of the preimage $\mathbb{H}'$ of $\mathbb{H}$ in the pro-universal tempered cover $S^\infty$ of $\mathcal{G}$.
- $\Delta_{\chi,\mathbb{H}}^{\text{temp}} := \Delta_{\chi}^{\text{temp}} \times \Delta_{\mathcal{G}}^{\text{temp}} \Delta_{\mathcal{H}}^{\text{temp}}$.
- Outer action of $G_K$ on $\Delta_{\chi,\mathcal{H}}^{\text{temp}}$.
Decomposition groups of subgraphs (IUTT1, cor. 2.3)

- $H'$: pro-connected component of the preimage of $H$ in $S^\infty$;
- $\hat{H}'$: "closure" of $H$ in $\hat{S}$;
- $\Delta_{\chi,H}^{temp} = \text{Stab}_{\Delta_{\chi,H}}(H')$; $\hat{\Delta}_{\chi,H} = \text{Stab}_{\hat{\Delta}_{\chi}}(\hat{H}')$;
- $\Pi_{\chi,H}^{temp} = \text{Stab}_{\Pi_{\chi,H}}(\Pi_{\chi,H})$; $\hat{\Pi}_{\chi,H} = \text{Stab}_{\hat{\Pi}_{\chi}}(\hat{H}')$;

\[
1 \rightarrow \Delta_{\chi,H}^{temp} \rightarrow \Pi_{\chi,H}^{temp} \rightarrow G_k \rightarrow 1
\]

\[
1 \rightarrow \hat{\Delta}_{\chi,H} \rightarrow \hat{\Pi}_{\chi,H} \rightarrow G_k \rightarrow 1
\]

Proposition

Let $I_{\chi} \subset \Delta_{\chi,H}^{temp}$ be an inertia group (of a cusp $\chi$) lying in a conjugate of $\Delta_{\chi,H}^{temp}$. Then $\chi$ abuts into $H$. 