Crelle's 473(1996), 207–222 © IF

On general local reciprocity maps

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In this paper we study abelian totally ramified p-extensions of a complete discrete valuation field with arbitrary residue field of characteristic $p > 0$ which is not separably p-closed. This is a generalization of the theory for the perfect residue field case which is exposed in [2]. At the same time this is in a certain sense a simplification of multidimensional local class field theory, since the theory in this paper provides in particular a description of abelian totally ramified extensions of higher dimensional local fields without using K-groups.

In the first section two reciprocity maps between the group

$$\mathrm{Gal}(L/F)^{\sim} = \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)\big)$$

of continuous homomorphisms from the profinite group $\mathrm{Gal}(\widetilde{F}/F)$ of the maximal unramified abelian p-extension to the Galois group $\mathrm{Gal}(L/F)$ of a totally ramified finite Galois p-extension L/F considered as a discrete group and the subquotient group

$$(U_{1,F} \cap N_{L\widetilde{F}/\widetilde{F}} U_{1,L\widetilde{F}})/N_{L/F}U_{1,L}$$

of the group of principal units of F are established, and their properties are investigated. In particular, the homomorphism

$$\Psi_{L/F} \colon (U_{1,F} \cap N_{L\widetilde{F}/\widetilde{F}} U_{1,L\widetilde{F}})/N_{L/F}U_{1,L} \to \big(\mathrm{Gal}(L/F)^{\mathrm{ab}}\big)^{\sim}$$

is surjective and its kernel is

$$(U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \cap N_{L\mathcal{F}/\mathcal{F}} U_{1,L\mathcal{F}})/N_{L/F}U_{1,L},$$

where $\mathcal{F}$ is any complete discrete valuation field which is an extension of F with ramification index being equal to 1 and with residue field being equal to the perfection of the residue field of F (Proposition (1.6)). If L/F is a cyclic extension, then this kernel is trivial and $\Psi_{L/F}$ is an isomorphism (Theorem (1.9)).

We apply this theorem in (1.10) to show that the norm groups $N_{L/F}U_{1,L}$ are in bijection with subextensions L/F of an abelian totally ramified p-extension.

Here one should mention a twenty-years-old work of Miki [14] (see Remark in (1.9)) a result of which in the case of cyclic extensions can be treated as a predecessor of the theory in this paper (I found that paper after the principal ideas in this paper had been formulated).

In (1.11) we consider other examples of abelian extensions for which the reciprocity map is an isomorphism. It remains a problem to understand for an arbitrary abelian totally ramified p-extension is $\Psi_{L/F}$ injective. The main obstruction is that in the case of imperfect residue field it isn't obvious at all that one can argue by induction on the degree of extensions contrary to the perfect residue field case or multidimensional class field theory. It is of interest to investigate a general problem when "norm – geometric" points of $A(F) \cap N_{L\widetilde{F}/\widetilde{F}} A(L\widetilde{F}) \cap N_{L\mathcal{F}/\mathcal{F}} A(L\mathcal{F})$ coincide with $N_{L/F} A(L)$ for an abelian variety A over F.

The second section deals with extensions for which a fixed prime element is a norm. We deduce from the theory of section 1 that the compositum of two such extensions is a totally ramified extension. Then an application of Theorem (1.10) provides an "elementary" proof of Theorem (2.2): for a complete discrete valuation field F with non-separably-p-closed residue field the norm group $N_{L/F} L^*$ is uniquely determined by an abelian totally ramified p-extension L/F.

The third section contains discussions on the existence theorem. In the general case of imperfect residue field one needs additional information in comparison with the perfect residue field case about the structure of norm subgroups. This is natural in view of the description of the norm groups in multidimensional class field theory ([4], [5], [6]). In this paper the existence theorem for cyclic extensions is established for the fields with small absolute ramification index ($< p-1$). It implies a connection between Witt vectors and cyclic p-extensions which has been earlier discovered by Kurihara [11] who employed a very different approach. This connection is explicitly described using the theory of fields of norms.

Thanks go to Christopher Hall for his remarks and corrections (added in 2020).

1. Reciprocity maps

Let F be a complete (or Henselian) discrete valuation field with a residue field $\overline{F}$ of characteristic $p > 0$. It will be assumed that $\overline{F}$ has a nontrivial separable p-extension. If $\overline{F}$ is separably p-closed, then class field theory of F is the limit of the theories for subfields F_α with non-separably-p-closed residue fields when F_α tends to F. Denote by $\widetilde{F}$ the maximal unramified abelian p-extension of F, i.e. the unramified extension corresponding to the maximal abelian p-extension $\overline{F}^{\mathrm{abp}}$ of the residue field $\overline{F}$. It is known that $\mathrm{Gal}(\overline{F}^{\mathrm{abp}}/\overline{F})$ is a free abelian profinite p-group on $\kappa = \dim_{\mathbb{F}_p} \overline{F}/\wp(\overline{F})$ generators, where $\wp(X) = X^p - X$. Then there is a non-canonical isomorphism $\mathrm{Gal}(\widetilde{F}/F) \simeq \prod_\kappa \mathbb{Z}_p$.

In fact for the theory exposed below one can take instead of $\widetilde{F}/F$ any its free profinite subextension $\hat{F}/F$.

Let U_F be the group of units of the ring of integers of F and let $U_{i,F}$ denote the subgroup of principal units $\equiv 1 \mod \pi_F^i$ with a prime element π_F of F. For an element θ of U_F by $\overline{\theta}$ we will denote its residue in $\overline{F}$.

1.1. Let L/F be a Galois totally ramified p-extension. Then $\mathrm{Gal}(L/F)$ can be identified with $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$, and $\mathrm{Gal}(\widetilde{L}/F)$ is isomorphic with $\mathrm{Gal}(\widetilde{L}/\widetilde{F}) \times \mathrm{Gal}(\widetilde{L}/L)$. Let $\mathrm{Gal}(L/F)^\sim = \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)\big)$ denote the group of continuous homomorphisms from the profinite group $\mathrm{Gal}(\widetilde{F}/F)$ which is a $\mathbb{Z}_p$-module ($a \cdot \sigma = \sigma^a$, $a \in \mathbb{Z}_p$) to the discrete $\mathbb{Z}_p$-module $\mathrm{Gal}(L/F)$. This group is isomorphic (non-canonically) with $\oplus_\kappa \mathrm{Gal}(L/F)$.

Now let L/F be of finite degree. Let $\chi \in \mathrm{Gal}(L/F)^\sim$ and Σ_χ be the fixed field of all $\tau_\varphi \in \mathrm{Gal}(\widetilde{L}/F)$, where $\tau_\varphi\big|_{\widetilde{F}} = \varphi, \tau_\varphi\big|_L = \chi(\varphi)$ and φ runs a topological $\mathbb{Z}_p$-basis of $\mathrm{Gal}(\widetilde{F}/F)$. Then $\widetilde{L}/\Sigma_\chi$ is unramified and Σ_χ/F is a totally ramified p-extension.

For a prime element π_χ of Σ_χ put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F} \pi_\chi N_{L/F} \pi_L^{-1} \mod N_{L/F} U_L,$$

where π_L is a prime element in L. We call $\Upsilon_{L/F}$ *a generalized Neukirch's map* as a generalization of constructions in [15].

1.2. Lemma. *The map* $\Upsilon_{L/F}\colon \operatorname{Gal}(L/F)^{\sim} \to U_F/N_{L/F}U_L$ *is well defined.*

Note that if $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\beta$ with $\beta \in U_{\widetilde{L}}$, then one can write $\beta = \theta\eta$ with $\theta \in U_L, \eta \in U_{1,\widetilde{L}}$ and then $\varepsilon' = N_{\widetilde{L}/\widetilde{F}}\eta \in U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ is uniquely defined mod $N_{L/F}U_{1,L}$. Thus, the quotient group $U_F \cap N_{\widetilde{L}/\widetilde{F}}U_{\widetilde{L}}/N_{L/F}U_L$ is mapped isomorphically onto $U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}/N_{L/F}U_{1,L}$ by $\varepsilon \to \varepsilon'$. Put

$$U(L/F) = U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}, \quad N(L/F) = N_{L/F}U_{1,L}$$

and denote the map $\operatorname{Gal}(L/F)^{\sim} \to U(L/F)/N(L/F)$ by the same notation $\Upsilon_{L/F}$.

1.3. Proposition.

(1) *Let L/F, L_1/F_1 be totally ramified Galois p-extensions, and F_1/F, L_1/L be totally ramified. Then the diagram*

$$\begin{array}{ccc}
\operatorname{Gal}(L_1/F_1)^{\sim} & \xrightarrow{\Upsilon_{L_1/F_1}} & U(L_1/F_1)/N(L_1/F_1) \\
\downarrow & & \downarrow N_{F_1/F} \\
\operatorname{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F)
\end{array}$$

is commutative, where the left vertical homomorphism is induced by the natural restrictions $\operatorname{Gal}(L_1/F_1) \to \operatorname{Gal}(L/F)$ and $\operatorname{Gal}(\widetilde{F}_1/F_1) \xrightarrow{\sim} \operatorname{Gal}(\widetilde{F}/F)$.

(2) *Let L/F be a totally ramified Galois p-extension, and let σ be an automorphism. Then the diagram*

$$\begin{array}{ccc}
\operatorname{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F) \\
\sigma^{\sim}\downarrow & & \downarrow \\
\operatorname{Gal}(\sigma L/\sigma F)^{\sim} & \xrightarrow{\Upsilon_{\sigma L/\sigma F}} & U(\sigma L/\sigma F)/N(\sigma L/\sigma F)
\end{array}$$

is commutative, where $(\sigma^{\sim}\chi)(\sigma\varphi\sigma^{-1}) = \sigma\chi(\varphi)\sigma^{-1}$.

(3) *Let F'/F $(F/F'$ resp.$)$ be an unramified extension of degree p^r. Let L/F $(L'/F'$ resp.$)$ be a totally ramified Galois p-extension. Let $L' = LF'$ $($resp. $L = L'F)$.*

Then the diagram

$$\begin{array}{ccc}
\operatorname{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F) \\
\downarrow & & \downarrow \\
\operatorname{Gal}(L'/F')^{\sim} & \xrightarrow{\Upsilon_{L'/F'}} & U(L'/F')/N(L'/F')
\end{array}$$

is commutative, where the left vertical homomorphism is multiplication by p^r $($resp. identity map$)$ and the right vertical homomorphism is induced by inclusion $($resp. by the norm map$)$.

(4) *Let F'/F $(F/F'$ resp.$)$ be an extension of degree p^r with purely inseparable extension of the residue fields of the same degree. Let L/F $(L'/F'$ resp.$)$ be a totally ramified Galois p-extension. Let $L' = LF'$ $($resp. $L = L'F)$. Then the diagram*

$$\begin{array}{ccc}
\operatorname{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F) \\
\downarrow & & \downarrow \\
\operatorname{Gal}(L'/F')^{\sim} & \xrightarrow{\Upsilon_{L'/F'}} & U(L'/F')/N(L'/F')
\end{array}$$

is commutative, where the left vertical homomorphism is identity map (resp. multiplication by p^r) and the right vertical homomorphism is induced by inclusion (resp. by the norm map).

Proof. For (1) apply the arguments of the proof of ([2], Proposition (1.8)) together with the following commutative diagram

$$\begin{array}{ccc} (U_{F_1}\cap N_{\widetilde{L}_1/\widetilde{F}_1}U_{\widetilde{L}_1})/N_{L_1/F_1}U_{L_1} & \longrightarrow & U(L_1/F_1)/N(L_1/F_1) \\ \big\downarrow N_{F_1/F} & & \big\downarrow N_{F_1/F} \\ (U_F\cap N_{\widetilde{L}/\widetilde{F}}U_{\widetilde{L}}/N_{L/F}U_L) & \longrightarrow & U(L/F)/N(L/F) \end{array}$$

□

1.4. Let $\mathfrak{F}$ be a complete discrete valuation field which is an extension of F such that $e(\mathfrak{F}|F)=1$ and the residue field of $\mathfrak{F}$ is the perfection of the residue field of $\widetilde{F}$, i.e. $=\cup_n \overline{\widetilde{F}}^{p^{-n}}$.

Let L/F be a finite totally ramified Galois p-extension. For $\sigma\in\mathrm{Gal}(L/F)$ put

$$c(\sigma)={\pi_L}^{-1}\sigma\pi_L \mod I(L|F),$$

where π_L is a prime element in L, and $I(L|F)$ is the subgroup of $U_{1,\widetilde{L}}$ generated by the elements $\varepsilon^{-1}\sigma(\varepsilon)$ with $\varepsilon\in U_{1,\mathfrak{L}},\ \sigma\in\mathrm{Gal}(L/F)$, $\mathfrak{L}=L\mathfrak{F}$. Then the sequence

$$1\to\mathrm{Gal}(L/F)^{\mathrm{ab}}\xrightarrow{c}U_{1,\widetilde{L}}/I(L|F)\xrightarrow{N_{\widetilde{L}/\widetilde{F}}}N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}\to 1$$

is exact (this follows from the case of perfect residue field, see [2]; [9], section 4; [10], (2.2)).

Now we introduce a reciprocity map acting in converse direction with respect to $\Upsilon_{L/F}$. Let $\varepsilon\in U_{1,F}\cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ and $\varphi\in\mathrm{Gal}(\widetilde{F}/F)$. Let $\eta\in U_{1,\widetilde{L}}$ be such that $N_{\widetilde{L}/\widetilde{F}}\eta=\varepsilon$. Since $N_{\widetilde{L}/\widetilde{F}}\big(\eta^{-1}\tilde{\varphi}(\eta)\big)=1$ for an extension $\tilde{\varphi}\in\mathrm{Gal}(\widetilde{L}/F)$ of φ, it follows that $\eta^{-1}\tilde{\varphi}(\eta)\equiv c(\sigma^{-1})$ for a suitable $\sigma\in\mathrm{Gal}(L/F)^{\mathrm{ab}}$, where π_L is a prime element in L. Set $\chi(\varphi)=\sigma$. Then $\chi(\varphi_1\varphi_2)=\sigma_1\sigma_2$, i.e. $\chi\in(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$. Put $\Psi_{L/F}(\varepsilon)=\chi$.

Lemma. *The map $\Psi_{L/F}\colon U(L/F)/N(L/F)\to(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$ is well defined and a homomorphism.*

Proof. If $N_{\widetilde{L}/\widetilde{F}}\rho=\varepsilon$, then for $\mu=\eta^{-1}\rho$ the element $\mu^{-1}\varphi(\mu)$ belongs to $I(L|F)$. If $\varepsilon=\varepsilon_1\varepsilon_2$, then one may assume $\eta=\eta_1\eta_2$, consequently $\sigma=\sigma_1\sigma_2$ in $\mathrm{Gal}(L/F)^{\mathrm{ab}}$. Thus, $\Psi_{L/F}(\varepsilon_1\varepsilon_2)=\Psi_{L/F}(\varepsilon_1)\Psi_{L/F}(\varepsilon_2)$. □

We call $\Psi_{L/F}$ *a generalized Hazewinkel's homomorphism* as a generalization of constructions in [9].

1.5. Proposition. *Let L/F be a Galois totally ramified p-extension. The composition of the map*

$$\Upsilon_{L/F}\colon\ \mathrm{Gal}(L/F)^{\sim}\to U(L/F)/N(L/F)$$

and the map

$$\Psi_{L/F}\colon U(L/F)/N(L/F)\to\big(\mathrm{Gal}(L/F)^{\mathrm{ab}}\big)^{\sim}$$

is the identity on $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^{\sim}$. Thus, the homomorphism $\Psi_{L/F}$ is surjective and $\ker\Upsilon_{L/F}=\mathrm{Gal}(L/L\cap F^{\mathrm{ab}})^{\sim}$.

Proof. Indeed, let $\pi_\chi = \pi_L \eta$ with $\eta \in U_{\widetilde{L}}$. Let $\varphi = \tilde{\varphi}\big|_{\widetilde{F}} \in \mathrm{Gal}(\widetilde{F}/F)$ with $\tilde{\varphi} \in \mathrm{Gal}(\widetilde{L}/L)$ and $\tau_\varphi \in \mathrm{Gal}(\widetilde{L}/F)$ be such that $\tau_\varphi\big|_{\widetilde{F}} = \varphi$, $\tau_\varphi\big|_L = \sigma = \chi(\varphi)$. Since the extension Σ_χ/F is totally ramified, one can write $\eta = \theta\eta_1$ with $\theta \in U_F$, $\eta_1 \in U_{1,\widetilde{L}}$. Then

$$\pi_L^{1-\sigma} = \eta^{\tau_\varphi - 1} \equiv \eta_1^{\tilde{\varphi}-1} \mod I(L|F)$$

and $N_{\widetilde{L}/\widetilde{F}}\eta_1 = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}(\theta\pi_L)^{-1}$. Therefore, χ regarded as an element of $\mathrm{Gal}(L/L\cap F^{\mathrm{ab}})^\sim$ coincides with $\Psi_{L/F}(\Upsilon_{L/F}(\chi))$. $\square$

Before we treat in (1.6), (1.7) relations of the reciprocity maps with reciprocity maps in the perfect residue field case, and in (1.8) — a list of equivalent additional properties for the maps $\Upsilon_{L/F}$ and $\Psi_{L/F}$ to be isomorphisms.

1.6. Assume that the residue field of F is not perfect. Denote by $\mathcal{F}$ a complete discrete valuation field which is an extension of F such that $e(\mathcal{F}|F) = 1$ and the residue field of $\mathcal{F}$ is the perfection $\overline{F}^{\mathrm{perf}} = \overline{F}^{p^{-n}}$ of the residue field of F ($\mathcal{F}$ isn't uniquely defined).

Let L/F be a totally ramified finite Galois p-extension. Put $\mathcal{L} = L\mathcal{F}$. The maps $\Upsilon_{\mathcal{L}/\mathcal{F}}$ and $\Psi_{\mathcal{L}/\mathcal{F}}$ (see [2]) are compatible with their descendants for L/F: the diagrams

$$\begin{array}{ccc} \mathrm{Gal}(L/F)^\sim & \xrightarrow{\Upsilon_{L/F}} & U(L/F)/N(L/F) \\ \downarrow & & \downarrow{\scriptstyle \lambda_{L/F}} \\ \mathrm{Gal}(\mathcal{L}/\mathcal{F})^\sim & \xrightarrow{\Upsilon_{\mathcal{L}/\mathcal{F}}} & U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} \end{array}$$

and ($\lambda_{L/F}$ is induced by the inclusion)

$$\begin{array}{ccc} U(L/F)/N(L/F) & \xrightarrow{\Psi_{L/F}} & (\mathrm{Gal}(L/F)^{\mathrm{ab}})^\sim \\ {\scriptstyle \lambda_{L/F}}\downarrow & & \downarrow \\ U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} & \xrightarrow{\Psi_{\mathcal{L}/\mathcal{F}}} & (\mathrm{Gal}(\mathcal{L}/\mathcal{F})^{\mathrm{ab}})^\sim \end{array}$$

are commutative.

Since $\Psi_{\mathcal{L}/\mathcal{F}}$ is injective, we deduce that $\lambda_{L/F}$ is surjective and

$$\ker \Psi_{L/F} = \ker \lambda_{L/F} = (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}})/N_{L/F}U_{1,L}.$$

In other words, we get

Proposition. *Let L/F be a totally ramified finite Galois p-extension. Then $\Psi_{L/F}$ induces an isomorphism*

$$(U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}})/(U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}) \to (\mathrm{Gal}(L/F)^{\mathrm{ab}})^\sim.$$

1.7. Put $N_\star(L/F) = U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}$.

For a Galois subextension M/F in L/F from the exact sequence

$$1 \to \mathrm{Gal}(L/M)^\sim \to \mathrm{Gal}(L/F)^\sim \to \mathrm{Gal}(M/F)^\sim \to 1$$

we obtain the following commutative diagram:

$$\begin{array}{ccccc}
U(L/M)/N_\star(L/M) & \xrightarrow{N^*_{M/F}} & U(L/F)/N_\star(L/F) & \longrightarrow & U(M/F)/N_\star(M/F) \\
\downarrow & & \downarrow & & \downarrow \\
U_{1,\mathcal{M}}/N_{\mathcal{L}/\mathcal{M}}U_{1,\mathcal{L}} & \xrightarrow{N^*_{\mathcal{M}/\mathcal{F}}} & U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} & \longrightarrow & U_{1,\mathcal{F}}/N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}}
\end{array}$$

We conclude that

(1) $$\lambda^{-1}_{L/F}(N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}}) = N_{M/F}(U(L/M))N_\star(L/F).$$

If the extension L/F is abelian, then the exact sequence above corresponds to the exact sequence

$$1 \to U_{1,\mathcal{M}}/N_{\mathcal{L}/\mathcal{M}}U_{1,\mathcal{L}} \xrightarrow{N^*_{\mathcal{M}/\mathcal{F}}} U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} \to U_{1,\mathcal{F}}/N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}} \to 1$$

In particular,

(2) $$\ker N_{\mathcal{M}/\mathcal{F}} \subseteq N_{\mathcal{L}/\mathcal{M}}U_{1,\mathcal{L}},$$

and

(3) $$U(L/F) \cap N_\star(M/F) = N_{M/F}(U(L/M))N_\star(L/F).$$

1.8. The previous considerations are useful in the proof of the following

Proposition. *The following properties of a totally ramified finite abelian p-extension L/F are equivalent:*

(1) $\Psi_{L/F}$ *and* $\Upsilon_{L/F}$ *are isomorphisms.*
(2) $\Psi_{L/F}$ *is a monomorphism.*
(3) $U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} = N_{L/F}U_{1,L}$.
(4) *for every* $\eta \in U_{1,\widetilde{L}}$, *if* $\eta^{\varphi-1} \in I(L|F)$ *for all* $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$ *then* $\eta^{\varphi-1} \in I(L|F)^{\varphi-1}$ *for all* $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$.
(5) $\Upsilon_{L/F}$ *is surjective.*
(6) $U(L/F)$ *coincides with the set of elements in the form* $N_{\Sigma_\chi/F}\pi_\chi/N_{L/F}\pi_L$, *where* Σ_χ, π_χ, π_L *are as in* (1.1) *and* χ *runs* $\mathrm{Gal}(L/F)^\sim$.

Proof. Thanks to Proposition (1.5) the properties (1), (2), (5) are equivalent. According to (1.6) the property (2) is equivalent to (3), and according to the description of the map $\Upsilon_{L/F}$ (5) is equivalent to (6).

Now we verify that (2) and (4) are equivalent. Let $\eta^{\varphi-1}$ belongs to $I(L|F)$ for all $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$. Then

$$\Psi_{L/F}(N_{\widetilde{L}/\widetilde{F}}\eta) = 1,$$

therefore by (2) $N_{\widetilde{L}/\widetilde{F}}\eta \in N_{L/F}U_{1,L}$ and $\eta^{\varphi-1} \in I(L|F)^{\varphi-1}$ for all $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$. Conversely, let $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta \in U(L/F)$ and $\Psi_{L/F}(\varepsilon) = 1$, then $\eta^{\varphi-1}$ belongs to $I(L|F)$ for all $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$, and hence (4) implies $\eta \in I(L|F)L_\varphi$, where L_φ is the fixed field with respect to φ in the completion of $\widetilde{L}$. We conclude that $\varepsilon \in N_{L_\varphi/F\cap L_\varphi}U_{1,L_\varphi}$. Induction on κ then leads to the desired $\varepsilon \in N(L/F)$. $\square$

Remark. Let L/F be an abelian totally ramified p-extension. Assume that $\Psi_{E/M}$ is an isomorphism for all $F \subseteq M \subseteq E \subseteq L$. Then $N(M/F) \cap U(E/F) = N_{M/F}(U(E/M))$. To see this, use (3) in (1.7).

1.9. Now we consider the following corollary of the previous theory.

Theorem. *Let L/F be a cyclic totally ramified p-extension. Then the map $\Upsilon_{L/F}$ is an isomorphism*

$$\operatorname{Gal}(L/F)^{\sim} \simeq (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} = U(L/F)/N(L/F)$$

and the homomorphism $\Psi_{L/F}$ is the inverse isomorphism.

Proof. The group $I(L|F)$ in this case consists of the elements $\varepsilon^{-1}\sigma\varepsilon$, where σ is a generator of $\operatorname{Gal}(L/F)$ and ε runs $U_{1,\widetilde{L}}$. For a $\varphi \in \operatorname{Gal}(\widetilde{L}/L)$ the group $U_{1,\widetilde{L}}$ is $(\varphi - 1)$-divisible, and, thus, $I(L|F)$ is. It remains to apply Proposition (1.8). □

Remark. Miki in [14] has shown without explicit introduction of reciprocity maps that for a totally ramified cyclic extension F'/F of degree m and for a finite abelian unramified extension E/F of exponent m the group

$$(F \cap N_{EF'/E}U_{EF'})/N_{F'/F}U_{F'}$$

is canonically isomorphic to the character group of $\operatorname{Gal}(E/F)$.

1.10. Theorem. *Let $L_1/F, L_2/F$ and their compositum be abelian totally ramified p-extensions. Then*

$$N_{L_2/F}U_{1,L_2} \subseteq N_{L_1/F}U_{1,L_1} \quad \text{if and only if} \quad L_2 \supseteq L_1.$$

Proof. Let M/F be a cyclic subextension in L_1/F. Then

$$N(L_2/F) \subseteq N(M/F) = N_\star(M/F).$$

Since

$$N_\star(L_2M/F) \subseteq U(L_2M/F) \cap N_\star(M/F),$$

we get

$$\begin{aligned} N_{L_2/F}(U(L_2M/L_2))N_\star(L_2M/F) &\subseteq U(L_2M/F) \cap N_\star(M/F) \\ &= N_{M/F}(U(L_2M/M))N_\star(L_2M/F) \end{aligned}$$

by (3) in (1.7). This inclusion and (1) in (1.7) show that

$$\lambda^{-1}_{L_2M/F}(N_{\mathcal{L}_2/\mathcal{F}}U_{1,\mathcal{L}_2}) \subseteq \lambda^{-1}_{L_2M/F}(N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}}).$$

From here we deduce that $M \subseteq L_2$. Thus, $L_2 \supseteq L_1$. □

1.11. In the general case of abelian totally ramified extensions it remains an open problem is $\Psi_{L/F}$ injective. We note that if $h_{L/F}$ is the Hasse-Herbrand function of an abelian totally ramified p-extension L/F, then the reciprocity isomorphism implies that $\Psi_{L/F}$ transforms

$$(U_{i,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}})/N_{L/F}U_{h_{L/F}(i),L}$$

isomorphically onto

$$\operatorname{Hom}_{\mathbb{Z}_p}\big(\operatorname{Gal}(\widetilde{F}/F), \operatorname{Gal}(L/F)_{h_{L/F}(i)}\big).$$

Now we prove

Proposition. *Let L/F be a totally ramified finite abelian p-extension. Assume that*

$$U_{j,F} \cap N_{L/F} U_{1,L} = N_{L/F} U_{h_{L/F}(j),L}$$

for any natural j. In addition, assume that there is a subextension M/F of L/F such that L/M is of degree p and $\Psi_{M/F}$ is an isomorphism.

Then $\Psi_{L/F}$ is an isomorphism.

Proof. According to Proposition (1.8) it suffices to verify that

$$U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}} U_{1,\mathcal{L}} = N_{L/F} U_{1,L},$$

where $\mathcal{F}$ is as in (1.6).

Let $\alpha \in U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}} U_{1,\mathcal{L}}$. By the assumption in the statement of the proposition we get $\alpha = N_{M/F}\beta$ for a $\beta \in U_{1,M}$. Then β belongs to $N_{\mathcal{L}/\mathcal{M}} U_{1,\mathcal{L}}$, since the kernel of $N_{\mathcal{M}/\mathcal{F}}$ is inside $N_{\mathcal{L}/\mathcal{M}} U_{1,\mathcal{L}}$ as easily follows from [2].

Let $v_F(\alpha - 1) = j$. According to the assumption in the proposition we may assume that $\beta \in U_{h_{M/F}(j),M}$ and $\alpha = N_{\widetilde{L}/\widetilde{F}}(\gamma)$ with $\gamma \in U_{h_{L/F}(j),\widetilde{L}}$.

Let π_L be a prime element in L, and $\pi_M = N_{L/M}\pi_L$. Let σ be a generator of $\mathrm{Gal}(L/M)$,

$$\frac{\sigma\pi_L}{\pi_L} = 1 + \theta_0 \pi_L^s + \dots$$

with θ_0 having a nonzero residue. Then it is well known that

$$\begin{aligned} N_{L/M}(1+\theta\pi_L^i) &= 1 + \theta^p \pi_M^i + \dots \qquad &&\text{for } i < s,\ \theta \in U_M \\ N_{L/M}(1+\theta\pi_L^s) &= 1 + (\theta^p - \theta_0^{p-1}\theta)\pi_M^s + \dots \qquad &&\text{for } \theta \in U_M \\ N_{L/M}(1+\theta\pi_L^{s+pi}) &= 1 - \theta_0^{p-1}\theta\pi_M^{s+i} + \dots \qquad &&\text{for } i > 0,\ \theta \in U_M. \end{aligned} \tag{4}$$

There are several cases to be considered.

Let $h_{M/F}(j) < s$. Then $h_{M/F}(j) = h_{L/F}(j)$, since $h_{L/M}(x) = x$ for $x \leqslant s$. Writing $\beta = 1 + \eta\pi_M^{h_{M/F}(j)}$ with η having a non-zero residue, and $\gamma = 1 + \rho\pi_L^{h_{M/F}(j)}$ with ρ having a non-zero residue, we deduce that $f(\overline{\eta}) = f(\overline{\rho})^p$, where $f(X)$ is the additive polynomial describing the action of the $N_{M/F}$ from $U_{h_{M/F}(j),M}/U_{h_{M/F}(j)+1,M}$ to $U_{j,F}/U_{j+1,F}$, and $f(X)^p$ is the additive polynomial describing the action of the $N_{\widetilde{L}/\widetilde{F}}$ from $U_{h_{L/F}(j),\widetilde{L}}/U_{h_{L/F}(j)+1,\widetilde{L}}$ to $U_{j,\widetilde{F}}/U_{j+1,\widetilde{F}}$. From here we deduce that $\overline{\eta}$ is a pth power in $\overline{\widetilde{F}}$, and, therefore in $\overline{F}$. Then, according to (4), β can be written as $\beta' N_{L/M}\delta$ with $\beta' \in U_{h(j)+1,M}$. Then we repeat the same argument for β'.

Let $h_{M/F}(j) = s$. Write $\beta = 1 + \eta\pi_M^{h_{M/F}(j)}$ with η having a non-zero residue. Since $\beta \in N_{\mathcal{L}/\mathcal{M}} U_{1,\mathcal{L}}$, we deduce from (4) that $\overline{\eta}\overline{\theta}_0^{-p} \in \wp(\overline{\widetilde{F}}^{\mathrm{perf}})$. Then $(\overline{\eta}\overline{\theta}_0^{-p})^{p^r} \in \wp(\overline{\widetilde{F}})$ for some $r \geqslant 0$, and hence, for $r = 0$. Thus, $\beta \in N_{\widetilde{L}/\widetilde{M}} U_{1,\widetilde{L}}$. Since by (2) in (1.7) β belongs also to $N_{\mathcal{L}/\mathcal{M}} U_{1,\mathcal{L}}$, one deduces from (1.8) and (1.9) that $\beta \in N_{L/M} U_{1,L}$.

Let $h_{M/F}(j) > s$, then $\beta \in N_{L/M} U_{1,L}$.

Finally we deduce that $\beta \in N_{L/M} U_{1,L}$. Thus, $\alpha \in N_{L/F} U_{1,L}$. □

Remark. Suppose we are in the case when the residue field of F is imperfect. By using this proposition one can show that $\Psi_{L/F}$ is an isomorphism if $\mathrm{Gal}(L/F)$ is the product of cyclic groups of order p and a cyclic group. Thus, the first unknown case is the case of an extension $L = M_1 M_2/F$, where M_i/F are cyclic extensions of degree p^2.

2. Extensions with a fixed prime element as a norm

2.1. Proposition. *Let F be a complete discrete valuation field with arbitrary residue field of characteristic p. Let L_1/F, L_2/F be abelian totally ramified p-extensions and $\pi \in N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$ for a prime element π of F. Then L_1L_2/F is a totally ramified extension.*

Proof. If $\overline{F}$ is algebraically p-closed, then the assertion is evident. If $\overline{F}$ is perfect and not algebraically p-closed, then L_1L_2/F is totally ramified by [2], (3.3). If $\overline{F}$ is imperfect, then let $\mathcal{F}$ be as in (1.6).

In the case of positive characteristic of F the field $\mathcal{F}$ can be chosen as a purely inseparable extension of F, and then $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$.

If F is of characteristic 0, then for any extension F' of F and for any $\overline{\theta} \in \overline{F}' \setminus \overline{F'}^p$ there exists $\alpha \in U_{F'}$ such that $\overline{\alpha} = \overline{\theta}$ and $L_1L_2F'(\alpha_1) \neq L_1L_2F'$ for $\alpha_1^p = \alpha$. Indeed, otherwise one would deduce that $U_{1,F'} \subseteq (L_1L_2F')^p$ which is impossible, since L_1L_2F'/F' is of finite degree and $U_{1,F'}/U_{1,F'}^p$ is of infinite order. Assume that $L_1F' \cap L_2F' = (L_1 \cap L_2)F'$, then if $L_1F'(\alpha_1) \cap L_2F'(\alpha_1) \neq (L_1F' \cap L_2F')(\alpha_1)$, one would have $M_1(\alpha_1) = M_2(\alpha_1)$ for a suitable extensions $M_1/(L_1 \cap L_2)F'$ in $L_1F'/(L_1 \cap L_2)F'$ and $M_2/(L_1 \cap L_2)F'$ in $L_2F'/(L_1 \cap L_2)F'$ of degree p. Then it would be $M_1(\alpha_1) = M_1M_2 \subseteq L_1L_2F'$, a contradiction. Thus, proceeding in this way, one can construct $\mathcal{F}/F$ with the property $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$ and $\overline{\mathcal{F}} = \overline{F}^{\mathrm{perf}}$.

Now, $L_1L_2\mathcal{F}/\mathcal{F}$ is totally ramified (as the residue field of $\mathcal{F}$ is perfect) of degree

$$\begin{aligned}
&|L_1L_2\mathcal{F} : (L_1 \cap L_2)\mathcal{F}|\,|(L_1 \cap L_2)\mathcal{F} : \mathcal{F}| \\
&= |L_1\mathcal{F} : (L_1 \cap L_2)\mathcal{F}|\,|L_2\mathcal{F} : (L_1 \cap L_2)\mathcal{F}|\,|(L_1 \cap L_2)\mathcal{F} : \mathcal{F}| \\
&= |L_1 : L_1 \cap L_2|\,|L_2 : L_1 \cap L_2|\,|L_1 \cap L_2 : F| = |L_1L_2 : F|.
\end{aligned}$$

Therefore L_1L_2/F is totally ramified. □

Remark. If the residue field of F is perfect, then under assumptions of the proposition $\pi \in N_{L_1L_2/F}U_{1,L_1L_2}$ (see [2], section 3). If the residue field is imperfect, this doesn't hold in general. Indeed, for a cyclic totally ramified extension L/F of degree p put $s(L|F) = v_L(\pi_L^{-1}\sigma\pi_L - 1)$ for a generator σ of the Galois group, where v_L if the surjective discrete valuation of L. Let L_1/F, L_2/F be totally ramified Galois extensions of degree p such that $p < s(L_1|F) \leqslant s(L_2|F)$. Assume that L_1L_2/F is totally ramified and $\pi \in N_{L_1L_2/F}(L_1L_2)^*$ is a prime element of F. Then $(1+\theta\pi)^p = 1 + \theta^p\pi^p + \dots$ belongs to $N_{L_1/F}U_{1,L_1}$ and $N_{L_2/F}U_{1,L_2}$ and for $\overline{\theta} \notin \overline{F}^p$ doesn't belong to $N_{L_1L_2/F}U_{1,L_1L_2}$, since $(N_{L_1L_2/F}U_{1,L_1L_2} \cap U_{p,L_1L_2})U_{p+1,L_1L_2}/U_{p+1,L_1L_2} = \overline{F}^{p^2}$. The prime element $\pi(1+\theta\pi)^p$ belongs to $N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$, and does not to $N_{L_1L_2/F}L_1L_2^*$.

2.2. Theorem. *Let F be a complete discrete valuation field with a residue field of characteristic p which isn't separably p-closed. Let L_1/F, L_2/F be totally ramified abelian p-extensions. Then $N_{L_1/F}L_1^* = N_{L_2/F}L_2^*$ if and only if $L_1 = L_2$.*

Proof. According to the previous proposition L_1L_2/F is totally ramified. Theorem (1.10) implies now that $L_1 = L_2$. □

Remark. A weaker assertion (for the case when the residue field if contained in an extension of fields of type $k((t_1))...((t_n))/k$ with a perfect not-p-closed field k) has been proved in [3] by using higher local class field theory. Note that if one replaces the words "totally ramified abelian p-extensions" by either "totally ramified abelian extensions", or by "abelian p-extensions", or by "totally ramified p-extensions", then the assertion of the theorem doesn't hold in general (see [3]).

3. On existence theorem

Let F be a complete discrete valuation field with residue field of characteristic p.

3.1. In the general case of imperfect residue field it seems difficult to describe norm subgroups of a cyclic totally ramified p-extensions of F (for the perfect case see [2], section 3).

For example, in the case of a Galois totally ramified extension L/F of degree p take a prime element π_L of L and $\pi_F = N_{L/F}\pi_L$, and let $\pi_L^{-1}\sigma\pi_L = 1+\theta_0\pi_L^s$ with θ_0 having a nonzero residue. Let v_F be the discrete valuation of F. Let $e_i(X) = X^p + a_{p-1}X^{p-1} + \dots + a_0$ be an irreducible polynomial of π_L^i over F for i prime to p. After some calculations one deduces that $m(i) = \min_{0\leqslant t<p} v_F(a_t) = v_F(a_j)$ with $ij \equiv -s \mod p$ and $a_j = is^{-1}\theta_0^{p-1}\pi_F^{i+s-(s+ij)/p} + \dots$. Moreover, one can show that there exists an element $\alpha = \pi_L^i + \dots \in L$ satisfying the equation $g_i(\alpha) = 0$, where $g_i(X) = X^p + b_jX^j + b_0$, $v_F(b_j) = v_F(a_j)$, $v_F(b_0) = i$ (see, for instance, [1]). This implies that

$$N_{L/F}(1-\theta\alpha) = 1 + b_j\theta^{p-j} + b_0\theta^p, \quad v_F(b_j) = m(i), \quad v_F(b_0) = i$$

for θ in the ring of integers of F. In the case of perfect residue field these formulas show that there is a polynomial $g_i(X)$ such that

$$1 + \theta\pi_F^i + g_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}$$

for any θ in the ring of integers of F (see [7], Chapter V section 3).

If the absolute ramification index of F is $\geqslant p-1$, this isn't the case for imperfect residue field: one can't expect that there is a polynomial $g_i(X)$ such that for all θ

$$1 + \theta^p\pi_F^i + g_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}.$$

Certainly, instead of this one can take an expression of the form

$$1 + \theta^{p^{n(i)}}\pi_F^i + h_i(\theta)\pi_F^s \in N_{L/F}U_{1,L}$$

with some $n(i)$ (even with $n(i) \leqslant 2$). As a direct generalization of the description of norm subgroups in the perfect residue field case (see [2], (3.1)), one would have expected the following: let π be a prime element of F. Then subgroups $\mathcal{N}$ in $U_{1,F}$ which are norm groups of cyclic totally ramified extension L/F of degree p with $\pi \in N_{L/F}L^*$ are characterized as (1) $\mathcal{N}$ is open; (2) for any $i > 0$ there exists a polynomial $f_i(X)$ with coefficients in the ring of integers $\mathcal{O}_F$ of F such that its residue $\overline{f}_i$ is nonzero $\overline{F}$-decomposable and $1 + f_i(\theta^{p^{n(i)}})\pi^i \subset \mathcal{N}$ for $\theta \in \mathcal{O}_F$; (3) for any $i > 0$ the image of $(U_{i,F} \cap \mathcal{N})U_{i+1,F}$ under the projection $U_{i,F} \to U_{i,F}/U_{i+1,F} \xrightarrow{\sim} \overline{F}$, $1+\theta\pi^i \to \overline{\theta}$, is equal to $p_i(\overline{F})$, where $p_i(X) = X^p$ for $i < s$, $p_i(X) = X$ for $i > s$, and $p_s(X) = X^p - \overline{\theta}_0^{p-1}X$.

However, there exist subgroups $\mathcal{N}$ satisfying these properties which *are not* norm subgroups. For example, for $e = 3$, $p = 3$, $s = 4$, and an imperfect residue field the subgroup $\mathcal{N} \subset U_{1,F}$ defined by the relations

$$\begin{aligned} 1 + \theta^p\pi^i &\in \mathcal{N}, \quad \text{for} \quad i < s, \quad \theta \in \mathcal{O}_F \\ 1 + (\theta^p - \theta_0^{p-1}\theta)\pi^s &\in \mathcal{N}, \quad \text{for} \quad i = s, \quad \theta \in \mathcal{O}_F \\ 1 + \theta\pi^i &\in \mathcal{N}, \quad \text{for} \quad i > s, \quad \theta \in \mathcal{O}_F \end{aligned}$$

isn't a norm subgroup of every cyclic extension L/F with $\pi \in N_{L/F}L^*$ (in this case $g_1(X) = X^3 + {\pi'}^3 X^2 + \pi'$, $v_F(\pi') = 1$).

If F is an n-dimensional local field, then one can deduce the following. If L/F is a cyclic totally ramified extension of degree p, then $\varepsilon \in N_{L/F}U_{1,L}$ if and only if for every choice of local parameters $t_{n-1},\dots,t_1$ in F the symbol $\{\varepsilon, t_{n-1},\dots,t_1\}$ belongs to $N_{L/F}K_n^{\mathrm{top}}(L)$. This way one obtains a description of the norm subgroups $N_{L/F}U_{1,L}$ from the existence theorem in higher-dimensional class field theory ([4], Theorem (5.2) and [5], Theorem (4.2)). However, this description is very unexplicit.

3.2. There is a complete description of the norm subgroups of cyclic totally ramified p-extensions when $p > 2$ and the absolute ramification index $e(F)$ is $< p-1$.

For the fields of characteristic 0 introduce a function

$$\mathcal{E}_{n,\pi_F}\colon \underbrace{W_n(\overline{F}) \oplus \dots \oplus W_n(\overline{F})}_{e_F \text{ times}} \to U_{1,F}/U_{1,F}^{p^n}$$

by the formula

$$\mathcal{E}_{n,\pi_F}\left((a_{0,j},\dots,a_{n-1,j})_{1\leqslant j\leqslant e_F}\right) = \prod_{0\leqslant i\leqslant n-1,\, 1\leqslant j\leqslant e_F} E(\tilde{a}_{i,j}^{p^{n-1-i}} \pi_F^j)^{p^i},$$

where $E(X) = \exp(X + X^p/p + X^{p^2}/p^2 + \dots)$ is the Artin–Hasse function, and $\tilde{a}_{i,j}$ is a lifting of $a_{i,j} \in \overline{F}$ in the ring of integers of an inertia subfield F_0 ($e(F_0) = 1$, $\overline{F}_0 = \overline{F}$) of F.

Theorem. *Let F be a complete discrete valuation field of characteristic 0 with residue field of characteristic p. Cyclic totally ramified extensions L/F of degree p^n, such that $\pi_F \in N_{L/F}L^*$, are in one-to-one correspondence with subgroups*

$$\mathcal{E}_{n,\pi_F}(\mathbf{F}W_n(\overline{F}) \oplus \dots \oplus \wp W_n(\overline{F})\mathbf{F}(a_{0,j},\dots,a_{n-1,j}) \oplus W_n(\overline{F}) \oplus \dots) U_{1,F}^{p^n}$$

in $U_{1,F}$, where $1 \leqslant j \leqslant e_F$, $(a_{0,j},\dots,a_{n-1,j})$ is invertible in $W_n(\overline{F})$, $\wp = \mathbf{F} - 1$, and $\mathbf{F}$ is the Frobenius map.

Proof. First, let L/F be a cyclic totally ramified extension of degree p^n. Let $\pi_F = N_{L/F}\pi_L$ with a prime element π_L of L. For a generator σ of $\mathrm{Gal}(L/F)$ put $s_l = v_L(\sigma^{p^l}(\pi_L)/\pi_L - 1)$, $0 \leqslant l \leqslant n-1$. The Eisenstein polynomial $e_i(X) = X^{p^n} + a_{p^n-1}X^{p^n-1} + \dots + a_0$ of π_L^i for $(i,p) = 1$ satisfies the property: $v_F(a_t) \geqslant s_1 + (n-1-v_p(t))e_F$, where v_p is the p-adic valuation. This implies that $N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}}$ coincides with

$$\mathcal{E}_{n,\pi_F}(\mathbf{F}W_n(\overline{F}^{\mathrm{abp}}) \oplus \dots \oplus W_n(\overline{F}^{\mathrm{abp}}) \oplus \dots),$$

where $W_n(\overline{F}^{\mathrm{abp}})$ stand at the places starting from the s_1th one.

Now since $N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}} \cap U_{1,F}/N_{L/F}U_{1,L}$ is isomorphic to $\oplus_\kappa \mathrm{Gal}(L/F)$ according to Theorem (1.9), one obtains that $N_{L/F}U_{1,L}$ is of the type described in the assertion of the theorem.

Second, for the case of perfect residue field $\overline{F}$ it follows from the existence theorem ([2], Theorem (3.5)) that any subgroup of $U_{1,F}$ of the type indicated in the assertion of the theorem is $N_{L/F}U_{1,L}$ for some cyclic totally ramified p-extension L/F with $\pi_F \in N_{L/F}L^*$.

Thus, it remains to treat the case of imperfect residue field. Let $\mathcal{F}$ be as in (1.6). Denote by N the subgroup in $U_{1,F}$ indicated in the assertion of the theorem. For an extension E/F with $e(E|F)=1$ denote by N_E the subgroup in the group of principal units of E of the form

$$\mathcal{E}_{n,\pi_F}(\mathbf{F}W_n(\overline{E})\oplus\dots\oplus\wp W_n(\overline{E})\mathbf{F}(a_{0,j},\dots,a_{n-1,j})\oplus W_n(\overline{E})\oplus\dots)U_{1,E}^{p^n}.$$

According to the previous considerations there exists a totally ramified p-extension $\mathcal{F}'/\mathcal{F}$ such that $N_{\mathcal{F}'/\mathcal{F}}U_{1,\mathcal{F}'}=N_{\mathcal{F}}$ and $\pi_F\in N_{\mathcal{F}'/\mathcal{F}}\mathcal{F}'^*$. In fact this extension $\mathcal{F}'/\mathcal{F}$ is defined over a finite extension of F, so it is sufficient without loss of generality to treat the case when $\mathcal{F}'=E'\mathcal{F}$, E'/E is a totally ramified cyclic extension of degree p and $E=F(\theta),\theta^p=\theta_0\in F$, $E\subseteq\mathcal{F}$.

One has $N_{E'/E}U_{1,E'}\subseteq N_{\mathcal{F}}\cap U_{1,E}$, and the description of $N_{\mathcal{F}}$ together with injectivity of the homomorphism $U_{1,E}/U_{1,E}^{p^n}\to U_{1,\mathcal{F}}/U_{1,\mathcal{F}}^{p^n}$ imply that $N_{\mathcal{F}}\cap U_{1,E}=N_E$. Therefore $N_{E'/E}U_{1,E'}=N_E$ by Theorem (1.9).

Denote the degree of the extension $F(\zeta_p)/F$ by l. There exists a prime element of $F(\zeta_p)$ such that $\pi_{F(\zeta_p)}^l=\pi_F$. Then the element $\pi_{F(\zeta_p)}$ belongs to $N_{E'(\zeta_p)/E(\zeta_p)}E'(\zeta_p)^*$. In addition, $\pi_{F(\zeta_p)}\in N_{\sigma E'(\zeta_p)/E(\zeta_p)}(\sigma E'(\zeta_p))^*$ for any imbedding σ of $E'(\zeta_p)$ in $E'(\zeta_p)^{\mathrm{alg}}$ over $F(\zeta_p)$. By Proposition (2.1) $E'(\zeta_p)\sigma E'(\zeta_p)/E(\zeta_p)$ is a totally ramified extension. Since the extension $E'(\zeta_p)/E$ is abelian, and the norm map $N_{E(\zeta_p)/E}$ maps $U_{1,E(\zeta_p)}$ onto $U_{1,E}$, it follows from (1.3) and (1.9) that the group $N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)}$ is equal to $N_{E(\zeta_p)/E}^{-1}(N_E)$.

$$\begin{array}{cccc} & F'' & & E'(\zeta_p) \\ F' & & E' & \\ & F(\zeta_p) & & E(\zeta_p) \\ F & & E & \end{array}$$

Keeping in mind the specific structure of N_E one obtains $\varepsilon^{\sigma-1}\in N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)}$ for any $\varepsilon\in N_{\widetilde{E'(\zeta_p)}/\widetilde{E(\zeta_p)}}U_{1,\widetilde{E'(\zeta_p)}}\cap U_{1,E(\zeta_p)}$. The last conclusion together with (1.3) and (1.9) imply that $E'(\zeta_p)/F(\zeta_p)$ is an abelian extension. It isn't cyclic, since otherwise easy calculations show that

$$\pi_F^{\sigma-1}=(\pi_{F(\zeta_p)}^l)^{\sigma-1}=N_{E'(\zeta_p)/E(\zeta_p)}(\pi^{\sigma-1})^l\neq 1,$$

(where $\pi_{F(\zeta_p)}=N_{E'(\zeta_p)/E(\zeta_p)}(\pi)$) for a generator σ of $\mathrm{Gal}(E'(\zeta_p)/F(\zeta_p)$ which is impossible. Hence, there exists a cyclic totally ramified extension $F''/F(\zeta_p)$ such that $F''E(\zeta_p)=E'(\zeta_p)$. We deduce that

$$N_{F''/F(\zeta_p)}U_{1,F''}\subseteq N_{E'(\zeta_p)/E(\zeta_p)}U_{1,E'(\zeta_p)}\cap F(\zeta_p)=N_{F(\zeta_p)/F}^{-1}(N),$$

and the inclusion can be replaced by equality.

Again, by (1.3) and (1.9) we conclude that F''/F is an abelian extension, and there exists a cyclic totally ramified extensions F'/F such that $F''=F'(\zeta_p)$. Finally $N_{F'/F}U_{1,F'}=N$, $\pi_F\in N_{F'/F}F'^*$ as desired. □

Remark. The correspondence between Witt vectors of length n and cyclic totally ramified extension of degree p^n for the case $e_F=1$ has been established by M. Kurihara ([11]): there exists an exact sequence

$$1\to H^1(F,\mathbb{Z}/p^n)_{nr}\to H^1(F,\mathbb{Z}/p^n)\to W_n(\overline{F})\to 1$$

with nice functorial properties (in this case any cyclic p-extension has separable residue field extension, see [13]). The approach of Kurihara is based on the study of the sheaf of the etale vanishing cycles on the special fiber of a smooth scheme over the ring of integers of F and of filtrations on Milnor's K-groups of local rings. If we take p as a prime element of F, then the exactness of the sequence above follows from (4.2).

3.3. One can ask, generalizing a question of Kurihara set in [11], what is an explicit description of the extension L/F corresponding to

$$N = \mathcal{E}_{n,\pi_F}(\mathbf{F}W_n(\overline{F}) \oplus \cdots \oplus \wp W_n(\overline{F})\mathbf{F}(a_{0,j}, \dots, a_{n-1,j}) \oplus W_n(\overline{F}) \oplus \dots)$$

according to Theorem (3.2).

The answer is known for $n = 1$:

$$L = F(\alpha) \quad \text{with} \quad \wp(\alpha) = \alpha^p - \alpha = (\tilde{a}_{0,j}^p \pi_F^j)^{-1}$$

(see e.g. [7], Chapter III section 2).

Now let $n > 1$. Then, first of all, it isn't true that L/F can be defined as a Witt extension. Some information can be extracted from the theory of fields of norms due to Fontaine and Wintenberger (see [8], [19], or [7] Chapter III section 5).

Consider a tower of fields $F_i = F_{i-1}(\pi_i)$ with $\pi_i^p = \pi_{i-1}$, $\pi_0 = \pi_F$. Let M be the union of all F_i. Then $\mathcal{M}/\mathcal{F}$ for $\mathcal{F}$ as in (2.2) and $\mathcal{M} = M\mathcal{F}$ is an arithmetically profinite extension (see [8]).

Denote by $\mathbf{M}$ the corresponding field of norms. The preimage $\mathbf{N}$ in $U_{1,\mathbf{M}}$ of N is equal to

$$\mathcal{E}_{n,\pi_\mathbf{M}}(\mathbf{F}W_n(\overline{F}^{\mathrm{abp}}) \oplus \cdots \oplus \wp W_n(\overline{F}^{\mathrm{abp}})\mathbf{F}(a_{0,j}, \dots, a_{n-1,j}) \oplus W_n(\overline{F}^{\mathrm{abp}}) \oplus \dots)U_{1,\mathbf{M}}^{p^n},$$

where $\pi_\mathbf{M} = (\pi_i)$ is a prime element of $\mathbf{M}$.

The group $\mathbf{N}$ coincides with $N_{\mathbf{M}'/\mathbf{M}}U_{1,\mathbf{M}'}$ for a cyclic extension

$$\mathbf{M}' = \mathbf{M}(\wp^{-1}((a_{0,j}^{p^n}, \dots, a_{n-1,j}^{p^n})^{-1}(\pi_\mathbf{M}^{-j}, 0, \dots))).$$

This has been proved for the case of quasi-finite residue field by Sekiguchi [17], the same arguments and p-class field theory of [2] provide the proof for the case of perfect residue field. Finally, the arguments similar to those in the proof of Theorem (4.2) show that in the case of imperfect residue field (when one can take $\mathcal{F}$ as a purely inseparable extension of F) the situation is the same.

$$\begin{array}{cccc} & M' & & \mathcal{M}' \\ M & & \mathcal{M} & \\ & & & \\ & F_i' & & \mathcal{F}_i' \\ F_i & & \mathcal{F}_i & \\ & & & \\ & F' & & \mathcal{F}' \\ F & & \mathcal{F} & \end{array}$$

This cyclic extension corresponds to a cyclic extension $\mathcal{M}'/\mathcal{M}$ of the same degree according to the general theory of fields of norms. Even more, by the theory of field of norms, see the proof of ([7], Chapter III, Theorem 5.7), it originates from an extension F_i'/F_i for a sufficiently large i.

The preimage of $N_{F_i'/F_i}U_{1,F_i'}$ in $U_{1,\mathbf{M}}$ coincides with $\mathbf{N}$, since class field theory is compatible with the theory of fields of norms (for the case of finite residue field see [12] or [7], Chapter IV section 6). Hence $N_{F_i'/F_i}U_{1,F_i'} = N_{F_i/F}^{-1}(N)$. By using similar arguments with ones of the proof of the theorem, one can deduce that F_i'/F_i originates from a cyclic extension F'/F and $N_{F'/F}U_{1,F'}$ coincides with N.

Note, that $F_i' = F_i(\wp^{-1}((\tilde{a}_{0,j}^{p^n}, \dots, \tilde{a}_{n-1,j}^{p^n})^{-1}(\pi_i^{-j}, 0, \dots)))$. In other words, extensions F_i'/F_i are Witt extensions for $i \geqslant i(n)$. For instance, $i(1) = 0$ (Artin–Schreier extensions in characteristic 0), and $i(2) = 1$ (see [18], section 3).

By using the previous description one can develop an analogue of Witt duality for complete discrete valuation fields of characteristic 0.

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