On the image of noncommutative local reciprocity map

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0. Introduction

First steps in the direction of an arithmetic noncommutative local class field theory were described in [2] as an attempt to find an arithmetic generalization of the classical abelian class field theory; see [3] for an exposition of its main features. In particular, [2] clarified and simplified the metabelian local class field theory of H. Koch and E. de Shalit [7], [8]. In the noncommutative local class field theory [2] a direct arithmetic description of Galois extensions of a fixed local field \( F \) is given by means of noncommutative reciprocity maps between the Galois group \( \text{Gal}(L/F) \) of a totally ramified arithmetically profinite Galois extension \( L/F \) and a certain subquotient of formal power series in one variable over the algebraic closure of the residue field of \( F \) (which, more precisely, is the completion of the maximal unramified extension of the field of norms of \( L/F \)). One of the reciprocity maps (see below for definitions) is

\[
N_{L/F}: \text{Gal}(L/F) \rightarrow U_{N(L/F)}^\circ / U_{N(L/F)}.
\]

This map is an injective 1-cocycle (the right hand side has a natural action of the Galois group). It is not surjective, and not a homomorphism in general. In the general case of nonabelian extensions, the description of the Galois group in this approach is given by objects related not only to the ground field \( F \) but to \( L \) as well.

To describe the image of the reciprocity map one can use a map from \( \text{Gal}(L/F) \) to \( U_{N(L/F)}^\circ / Y_{L/F} \) induced by \( N_{L/F} \), where \( Y_{L/F} \) is a certain subgroup of \( U_{N(L/F)}^\circ \) containing \( U_{N(L/F)} \), such that the induced map is bijective. A key problem is to obtain as much information as possible about the subgroup \( Y_{L/F} \). Then via the reciprocity map \( N_{L/F} \) this information translates into a description of the Galois group of \( L/F \).

In this short note we suggest a new definition of certain maps \( f_i \) (see section 2) for regular extensions \( L/F \). This provides more information on the submodule \( Y_{L/F} \).

Needless to say, this arithmetic approach to noncommutative local class field theory is very different from the Langlands approach, which is somehow less arithmetic. From a general point of view it should be quite difficult to get a sufficiently explicit description of \( Y_{L/F} \) for an arbitrary class of extensions \( L/F \). There is a nice explicit description in the case of metabelian extensions, see [2],[7],[8]. It is expected there is a good explicit description in the case of \( p \)-adic Lie extensions, on the basis of [2] and this work. This may be of use for the local noncommutative Iwasawa theory.

We will assume that the reader has a good knowledge of basic results on local fields, as given for example in [4, Ch.III–IV].
1. The abelian case: interpretation

We start with a brief description of an interpretation of the abelian reciprocity maps, since it is this interpretation which leads to the construction of noncommutative reciprocity maps.

Let $F$ be a local field with finite residue field whose characteristic is $p$. Denote by $F^{ur}$ the maximal unramified extension of $F$ in a fixed completion of a separable closure of $F$ and denote by $\mathcal{F}$ be the completion of $F^{ur}$.

Now we briefly present two (abelian) local reciprocity maps: geometric and arithmetic. Each of them uses the fact that for a finite Galois extension $L/F$ the homomorphism

$$\text{Gal}(L/F) \rightarrow \ker N_{L/F}(V(L/F)), \quad \sigma \mapsto \pi^\sigma - 1$$

is surjective with the kernel being the derived group of the Galois group. Here $V(L/F)$ is the augmentation subgroup generated by elements $u^\sigma - 1$ with $u \in U_L, \sigma \in \text{Gal}(L/F)$, and $\pi$ is any prime element of $L$. For a noncommutative generalization of this, see the first assertion of the Theorem below.

First, we give a description of the geometric reciprocity map. Let $L/F$ be a finite Galois extension. Viewing all objects with respect to the pro-algebraic Zariski topology [10] one has a commutative diagram

$$
\begin{array}{cccccccc}
1 & \longrightarrow & \pi_1(U_L) & \longrightarrow & U_L & \longrightarrow & 1 \\
\downarrow & & \downarrow N_{L/F} & & \downarrow N_{L/F} & & \downarrow N_{L/F} \\
1 & \longrightarrow & \pi_1(U_{\mathcal{F}}) & \longrightarrow & U_{\mathcal{F}} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1 & & 1 \\
\end{array}
$$

where $U_L$ and $U_{\mathcal{F}}$ are the universal covering spaces of $U_L$ and $U_{\mathcal{F}}$. Applying the snake lemma, one has a map

$$\sigma \mapsto \pi^\sigma - 1 \in U_L \longrightarrow N_{L/\mathcal{F}}(\alpha^{-1}(\pi^\sigma - 1)) \in \pi_1(U_{\mathcal{F}})/N_{L/\mathcal{F}}\pi_1(U_L)$$

which is the geometric reciprocity homomorphism (this is more or less straightforward from [10]).

For a separable extension $L$ of $F$ put $L^{ur} = LF^{ur}, L = L\mathcal{F}$. To define the arithmetic reciprocity map, let $L$ be a finite totally ramified Galois extension of $F$. Let $\varphi$ be an element of the absolute Galois group of $F$ such that its restriction to $F^{ur}$ is the Frobenius automorphism of $F$. Denote by the same notation the continuous extension of $\varphi$ to the completion of the maximal separable extension of $F$. Let $\pi$ be a prime element of $L$.

There is a commutative diagram

$$
\begin{array}{cccccccc}
1 & \longrightarrow & U_L & \longrightarrow & U_L & \longrightarrow & 1 \\
\downarrow N_{L/F} & & \downarrow N_{L/F} & & \downarrow N_{L/F} \\
1 & \longrightarrow & U_F & \longrightarrow & U_{\mathcal{F}} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1 & & 1 \\
\end{array}
$$

and, applying the snake lemma, one has a map

$$\sigma \mapsto \pi^\sigma - 1 \in U_L \longrightarrow N_{L/\mathcal{F}}((1 - \varphi)^{-1}(\pi^\sigma - 1)) \in U_F/N_{L/F}U_L$$
which is the arithmetic local reciprocity homomorphism [5], [6], [9], [4]. The equation
\[ u^{1-\varphi} = \pi^{\sigma-1} \]
plays a fundamental role for the arithmetic reciprocity homomorphism.

Of course, the geometric reciprocity homomorphism can be viewed as the projective limit of the arithmetic reciprocity homomorphisms.

2. The reciprocity map \( N_{L/F} \)

We present one of noncommutative reciprocity maps originally defined in [2].

Denote by \( F^\varphi \) the fixed subfield of \( \varphi \) in the separable closure of \( F \). Let \( L/F \) be a Galois arithmetically profinite extension which is infinite. We will suppose throughout the paper that \( L \subset F^\varphi \). For the theory of fields of norms of arithmetically profinite extensions see [11] and [4, Ch. III sect. 5]. The field of norms \( N(L/F) \) of the extension \( L/F \) is a local field of characteristic \( p \) with residue field isomorphic to the residue field of \( F \).

Denote by \( X \) the norm compatible sequence of prime elements of finite subextensions of \( F \) in \( L \) which is the part of the unique norm compatible sequence of prime elements in finite extensions of \( F \) in \( F^\varphi \) (for its existence and uniqueness see the first section of [8]).

Denote by \( N(L/F) \) the completion of the maximal unramified extension of the field \( N(L/F) \). Denote by \( U^0_{N(L/F)} \) the subgroup of those elements of \( U_{N(L/F)} \) whose \( \mathcal{F} \)-component belongs to \( U_F \).

The reciprocity map \( N_{L/F} \) is defined as
\[ N_{L/F} : \text{Gal}(L/F) \longrightarrow U^0_{N(L/F)}/U_{N(L/F)}, \quad N_{L/F}(\sigma) = U \mod U_{N(L/F)}, \]
where \( U \in U_{N(L/F)} \) satisfies the (quite similar to the above) equation
\[ U^{1-\varphi} = X^{\sigma-1}. \]
It was shown in [2] that the ground \( \mathcal{F} \)-component of \( N_{L/F} \) equals the arithmetic reciprocity map described above, so \( N_{L/F} \) is indeed a genuine extension of the abelian reciprocity map.

Fix a tower of subfields \( F = E_0 = E_1 = E_2 = \ldots \), such that \( L = \cup E_i \), \( E_i/F \) is a Galois extension, and \( E_i/E_{i-1} \) is cyclic of degree \( p \) for \( i > 1 \) and \( E_1/E_0 \) is cyclic of degree relatively prime to \( p \). Let \( \sigma_i \) be an element of \( \text{Gal}(L/E) \) whose restriction to \( E_i \) is a generator of \( \text{Gal}(E_i/E_{i-1}) \). Denote by \( v_{E_i} \) the discrete valuation of \( E_i = E_i/L \). Put \( s_i = v_{E_i}(\pi_{E_i}^{p-1} - 1) \) where \( \pi_{E_i} \) is a prime element of \( E_i \).

The group \( U^0_{N(L/F)} \) contains a subgroup \( Y_{L/F} \) (which contains \( U_{N(L/F)} \)) such that the reciprocity map \( N_{L/F} \) induces a bijection between \( \text{Gal}(L/F) \) and \( U^0_{N(L/F)}/Y_{L/F} \), see [2, Th.2]. To get more information on \( Y_{L/F} \) and its more explicit description, [2] uses certain liftings
\[ f_i : U^0_{E_i} \longrightarrow U_{N(L/E_i)} \longrightarrow U_{N(L/F)}. \]
This is a central part of the noncommutative class field theory, and the better the description of \( f_i \), the more information one obtains about the Galois extensions. Liftings \( f_i \) were defined in [2, Def. 3–4] by using arbitrary topological \( \mathbb{Z}_p \)-generators of \( U^0_{E_i} \).

**Definition.** Call an extension \( L/F \) regular if \( L \setminus \mathcal{F} \) contains no primitive \( p \)th root. In positive characteristic every extension is regular. In characteristic zero \( L/F \) is regular if and only if the extension \( F(\zeta_p)/F \) is unramified or the extension \( L(\zeta_p)/L \) is not unramified. In particular, if \( E_1 = E_0 \) then \( L/F \) is regular.
Now we make a correction for the paragraph standing between Definitions 3 and Definition 3' in the published version of [2]. The statement there holds for regular extensions. Indeed, let \( F \) be of characteristic zero. If a primitive \( p \)th root of unity \( \zeta_p \) equals \( \omega^{\sigma_i} \) with \( \sigma_i \in U_{E_k} \), then \( N_{E_k/F}(\zeta_p) = 1 \). Hence, if \( E_1 \setminus E_0 \) contains no primitive \( p \)th root of unity (i.e. \( L/F \) is regular), then so does \( U_{E_1}^{\sigma_i} \). All the assertions of [2] following Def. 3 hold for regular Galois arithmetically profinite extensions.

In the general case of (non-regular) extensions the group \( U_{E_1}^{\sigma_i} \) may have a nontrivial \( p \)-torsion (for example, if \( E_0 = \mathbb{Q}_p \) and \( E_1 = \mathbb{Q}_p(\zeta_p) \)). It is not clear at the moment how to define the corresponding map \( f_1 \) for non-regular extensions.

Below we give a new definition of \( f_1 \) for regular Galois arithmetically profinite extensions \( L/F \), \( L \subset F^\times \).

3. Splitting exact sequences

The following theorem leads to a new definition of liftings \( f_i \).

For submodules \( M_i \) of \( U_{E_k} \) denote by \( \prod M_i \) their product.

Denote \( E = E_k, E' = E_{k+1}, E' = E_{k+1}, F = \mathbb{Q}_p(\zeta_p) \). Denote \( \sigma = \sigma_{k+1} \).

Recall that in the abelian class field theory an important role is played by the following exact sequence

\[
1 \longrightarrow T \longrightarrow U_{E'}/U_{E'}^{\sigma_i-1} \longrightarrow U_{E} \longrightarrow 1.
\]

Here \( T \) is the isomorphic image of \( \text{Gal}(E'/E) \) in \( U_{E'}/U_{E'}^{\sigma_i-1} \) with respect to the homomorphism

\[
\text{Gal}(E'/E) \longrightarrow U_{E'}/U_{E'}^{\sigma_i-1}, \quad \rho \longmapsto \pi_{E_{k+1}/E'}^{\rho-1}\pi_{E_{k+1}^{\sigma_i}}^{E_{k+1}/E_{k+1}}.
\]

see [4, Ch.IV (1.7)].

**Theorem.** Fix \( k \geq 1 \). Assume that \( L/F \) is a regular extension.

Denote by \( T' \) the intersection of \( T \) with \( \prod_{i \leq k} U_{E_i}^{\sigma_i-1}/U_{E_i}^{\sigma_i-1} \). We have an exact sequence

\[
1 \longrightarrow T' \longrightarrow (\prod_{i \leq k} U_{E_i}^{\sigma_i-1})/U_{E_i}^{\sigma_i-1} \longrightarrow (\prod_{i \leq k} U_{E_i}^{\sigma_i-1})/U_{E_i}^{\sigma_i-1} \longrightarrow 1.
\]

The sequence splits by a (not uniquely determined in general) homomorphism

\[
f: \prod_{i \leq k} U_{E_i}^{\sigma_i-1} \longrightarrow (\prod_{i \leq k} U_{E_i}^{\sigma_i-1})/U_{E_i}^{\sigma_i-1}.
\]

**Proof.** It is convenient to divide it into several parts.

1. The product of modules \( \prod_{i \leq k} U_{E_i}^{\sigma_i-1} \) is a closed \( \mathbb{Z}_p \)-submodule of \( U_{1,E} \). Let \( \lambda_j \) be a system of topological multiplicative generators of the topological \( \mathbb{Z}_p \)-module \( \prod_{i \leq k} U_{E_i}^{\sigma_i-1} \) which satisfy the following property: if the torsion of this group is nontrivial, it includes \( \lambda_j \) of order \( p^m \), and the rest of \( \lambda_j \) are topologically independent over \( \mathbb{Z}_p \).

Define a map \( f \) on the topological generators \( \lambda_j \) as

\[
f(\lambda_j) = u_j U_{E_i}^{\sigma_i-1}
\]

where \( u_j \) is any element of \( (\prod_{i \leq k+1} U_{E_i}^{\sigma_i-1}) \) whose norm equals \( \lambda_j \). We will prove by the end of the fifth part that \( f(\lambda_j) = \zeta \in U_{E_i}^{\sigma_i-1} \). Hence we can extend \( f \) to a homomorphism \( f: \prod_{i \leq k} U_{E_i}^{\sigma_i-1} \longrightarrow \prod_{i \leq k+1} U_{E_i}^{\sigma_i-1}/U_{E_i}^{\sigma_i-1} \) which is a section of the exact sequence in the theorem.
Suppose that $m > 0$, i.e. $\lambda_x$, different from 1, is in the system of the generators. As discussed at the end of the previous section, if $k = 1$ and $L/F$ is a regular extension then $m = 0$. Hence $k > 1$.

We claim that then $s_{k+1}$ (defined in section 2) is prime to $p$. This will be proved by the end of the fourth part.

2. By [4, Ch.III (2.3)] we know that if $s_{k+1}$ is divisible by $p$ then $s_{k+1} = p(e(E_k)/(p - 1))$, a primitive $p$th root lies in $E_k$ and there is a prime element $\pi_k$ of $E_k$ such that $E_{k+1} = E_k(\sqrt[p]{\pi_k})$. Using [4, Ch.II Prop. 4.5] we deduce that $s_i$ are divisible by $p$ for $2 \leq i \leq k + 1$. So all the ramification breaks $s_i$, $2 \leq i \leq k + 1$ their maximal possible values. Using local class field theory and looking at the norm group of $E_k/E_1$ it is easy to see that $E_k/E_1$ is a cyclic extension (see, e.g. [1, Prop. 1.5]). Then $\sigma_{E_k}$ is a generator of its Galois group. Recall that we assume that the torsion element belongs to the system of generators of $\prod U_{E_i}^{\pi_i}$. Hence a primitive $p$th root $\zeta_p$ can be written as $u_1^{s_1^{-1}}u_2^{s^{-1}}$ with $u_i \in U_{E_i}$. We will show by the end of the fourth part that this leads to a contradiction; then $s_{k+1}$ is prime to $p$.

3. Denote by $v$ the discrete valuation of $E$ and let $\pi$ be a prime element of $E$. To get a contradiction, choose $u_1$ with maximal possible value of $v(u_1 - 1)$ such that $\zeta_p = u_1^{s_1^{-1}}u_2^{s^{-1}}$. We will show that we can increase the value $v(u_1 - 1)$, and this gives a contradiction.

Using the description of the norm map in [4, Ch.III sect.1] we deduce that

$$
\pi^{s_1} = \theta_1 \pi + \text{terms of higher order},
$$

$$
\pi^{s_1^{-1}} = 1 + \theta_2 \pi(E_2)/(p-1) + \text{terms of higher order},
$$

with non-zero multiplicative representatives $\theta_1, \theta_2$ is a primitive $l$th root.

The Galois group of $E/F$ is the semi-product of cyclic groups of order $l = |E_1 : E_0|$ and $|E_k : E_1|$. Let $R$ be the fixed field of the first group. Then $\sigma_1|_E$ as a generator of the Galois group of $E/R$. Denote $\mathcal{R} = R\mathcal{E}$.

Let $\theta$ run through non-zero multiplicative representatives. In the first choice of representatives in $U_{E_i}/U_{E_{i+1}}$ of the group of principal units of $E$ we can include in it units $1 + \theta \pi_R^i$ where $\pi_R$ is a prime element of $R$. Note that $\sigma_1$ acts trivially on such elements. In addition,

$$(1 + \theta \pi_i)^{s_1} = 1 + \theta (\theta_i - 1) \pi + \text{terms of higher order}, \quad \text{if (i, l) = 1}.$$

In the second choice of topological generators of the group of principal units of $E$ take elements $1 + \theta \pi_i$, $(i, p) = 1$, $i < pe(E)/(p - 1)$ and an appropriate element $1 + \theta \pi^{pe(E)/(p-1)}$ (see, e.g. [4, Ch.I sect.6]). We get

$$(1 + \theta \pi_i)^{s_2^{-1}} = 1 + i\theta \pi_i^{i + c(E_2)/(p-1) + \text{terms of higher order}}, \quad \text{if (i, p) = 1}.$$

4. From the description of the behaviour of the map $x \mapsto x^p$ on the group of principal units (see, e.g., [4, Ch.I sect.5]) we deduce the following. If for some $r \geq 0$ the element $((1 + \theta \pi_i)^{pr})^{s_2^{-1}}$ with $(i, p) = 1$ is not closer to 1 than $\zeta_p$, then

$$(1 + \theta \pi_i)^{pr} = 1 + (i\theta \pi_i)^{pr (i + c(E_2)/(p-1)) + \text{terms of higher order}}, \quad (i, p) = 1.$$

Since $\zeta_p \in E_1, l$ divides $e(E_2)/(p - 1)$. From the previous description of the action of $\sigma_1$ we deduce that $v(u_1^{s_1^{-1}} - 1) = v(\zeta_p - 1)$ does not hold. Using the description of the action of $\sigma_2$ and observing that $\pi^{pr (i + c(E_2)/(p - 1))} = \pi_k^{pr - 2c(E_2)/(p - 1)} = v(\zeta_p - 1)$ for $r \geq 0$ implies $p$ divides $i$, we also deduce that $v(u_1^{s_1^{-1}} - 1) = v(\zeta_p - 1)$ does not hold.

Hence $v(u_1^{s_1^{-1}} - 1) = v(u_2^{s_2^{-1}} - 1) < v(\zeta_p - 1)$ and $u_2^{s_2^{-1}} = 1 + (i\theta \pi_i)^{pr (i + c(E_2)/(p-1)) + \text{terms of higher order}}$ for some $r \geq 0$. Denote $j = p^r (i + e(E_2)/(p - 1))$. Then $u_1^{s_1^{-1}}$ must start with
1 - (i\theta \beta) y^r \pi^j$, and hence $j$ is not divisible by $l$. Due to the choice of $u_1$ we can assume that when it is presented as the product of the first choice of representatives in the group of principal units of $E$, that product does not contain elements from $R$. Therefore $u_1 = w^{\sigma_1 - 1} u_2', v(u_1' - 1) > v(u_1 - 1)$, where $w = (1 + \eta \pi^r \eta)' \pi^r \equiv -\theta \pi^r (\theta_1' - 1)^{-1} \mod \pi$ and $w^{\sigma_2 - 1} = 1 - (i\theta \beta) y^r (\theta_1' - 1)^{-1} \pi^r + \text{terms of higher order}$.

Now $\zeta = u_1^{\sigma_1 - 1} u_2^{\sigma_2 - 1} = u_1^{\sigma_1 - 1} u_2^{\sigma_2 - 1}$ where $u_2 = u_2 w^{\sigma_1 - 1} \pi^r$. Here $z = 1$ is $E_k/E_0$ is abelian and $z = (w^{\sigma_1 \sigma_2})^{1+\ldots+\sigma_2 - 2} \in U_E$ where $(\sigma_1^{-1} \sigma_2) \in E \mod \pi$, $r > 1$, otherwise. Since $v(u_1' - 1) > v(u_1 - 1)$, we get a contradiction.

Thus, $s_{k+1}$ is prime to $p$.

5. Now, we argue similarly to the proof of a part of [2, Lemma 3]. Denote $\beta_a = u_+^m \rho$. We aim to show that $\beta_a \in U_{E'}^{\sigma_1 - 1}$. We get $N_{E'/E} \beta_a = 1$, hence $\beta_a$ can be written as $\pi^r w^{\sigma_1 - 1} \mod \rho \in \text{Gal}(E'/E)$, $u \in U_{E'}$, $\pi E'$ a prime element of $E'$. We shall show that $\rho = 1$. Then $w^{\sigma_1 - 1} \mod \rho$, as desired.

Find a unit $\delta$ in $E'$ such that $\delta^{1-\varphi} = u_+^{\sigma_1 - 1} \mod \rho$. Then, as briefly discussed in section 1, the reciprocity homomorphism for $E'/E$ maps $\rho$ to $N_{E'/E} \delta \mod \rho$; for more detail see [4, Ch.IV sect.3]. If $\varepsilon = \frac{N_{E'/E} \delta}{\rho}$ belongs to $E$, then the image of $\rho$ belongs to $N_{E'/E} U_{E'}$, and hence, since the reciprocity homomorphism is injective for abelian extensions, $\rho = 1$. If $\varepsilon$ does not belong to $E$, then, since $\varepsilon \in E$, we can write $\varphi^p = a^p \omega$ where $a \in U_E$ and $\omega \in U_{E'}$ is a $p$-primary element (i.e. the extension $E(\sqrt[p]{\omega})/E$ is unramified of degree $p$) Since $s_{k+1}$ is prime to $p$, we have $s_{k+1} < v(\omega - 1) = pe(E)/(p-1)$. Properties of the norm map (see e.g. [4, Ch. III sect. 1]) imply that $\omega \in N_{E'/E} U_{E'}$. Therefore the image of $\rho$, which is the class of $\varepsilon \varphi$, belongs to $N_{E'/E} U_{E'}$. Thus, $\rho = 1$, as desired.

Remark 1. The sequence $1 \rightarrow T \rightarrow U_{E'}/U_{E'}^{\sigma_1 - 1} \xrightarrow{N_{E'/E}} U_{E'} \rightarrow 1$ does not split if and only if $s_{k+1}$ is divisible by $p$ (i.e. the extension $E_{k+1}/E_k$ is not of Artin–Schreier type). This follows from the fifth part of the proof of the previous theorem.

Remark 2. $T' = \{1\}$ if and only if the extension $E'/F$ is abelian; in this case the splitting $f$ is uniquely determined.

Remark 3. The sequence $1 \rightarrow T' \rightarrow \left( \prod_{i<k+1} U_{E_i}^{\sigma_1 - 1} \right) / U_E^{\sigma_1 - 1} \xrightarrow{N_{E'/E}} \prod_{i<k+1} U_{E_i}^{\sigma_1 - 1} \rightarrow 1$ splits in the category of $\mathbb{Z}_p$-modules, but not necessarily in the category of pro-algebraic modules (see also Remark 5).

4. A new definition of $f_i$ and $Y_L/F$

We assume in this section that the reader has a good knowledge of [2].

Definition. Using the previous theorem, we introduce homomorphisms ($k \geq 1$)

$$h_k: \prod_{1 \leq i \leq k} U_{E_i}^{\sigma_1 - 1} \rightarrow \left( \prod_{1 \leq i \leq k+1} U_{E_i}^{\sigma_1 - 1} \right) / U_{E_{k+1}}^{\sigma_1 - 1}.$$  

Set $X_i = U_{E_i}^{\sigma_1 - 1}$. Let $g_k: \prod_{1 \leq i \leq k} U_{E_i}^{\sigma_1 - 1} \rightarrow \prod_{1 \leq i \leq k+1} U_{E_i}^{\sigma_1 - 1}$ be any map such that $h_k = g_k \mod U_{E_{k+1}}^{\sigma_1 - 1}$.  

Define
\[ f_i: X_i \rightarrow U_{N(L/E)} \rightarrow U_{N(L/F)} \]
as any map such that its \( E_j \)-component for \( j > i \) coincides with \((g_{j-1} \circ \cdots \circ g_i)|_{X_i}\).

This definition of \( f_i \), since it comes from the splitting homomorphisms in the previous theorem, is more functorial than that in [2].

With this choice of \( f_i \), [2, Lemma 4] holds for all regular extensions. Denote by \( Z_i \) the image of \( f_i \). Set \( Z_{L/F} = Z_{L/F}(\{E_i, f_i\}) = \left\{ \prod z^{(i)}: z^{(i)} \in Z_i \right\} \).

Define \( Y_{L/F} = \{ y \in U_{N(L/F)}: y^{1-\varphi} \in Z_{L/F} \}. \)

As in [2], the map \( 1 - \varphi \) induces an isomorphism between the group \( U_{N(L/F)}^0/Y_{L/F} \) and group \( \ker N_{L/F}/Z_{L/F} \).

The following theorem is proved exactly in the same way as [2, Th. 1 and Th. 2].

**Theorem.** Let \( L/F \) be a good Galois arithmetically profinite extension.

The map \( \text{Gal}(L/F) \rightarrow \ker N_{L/F}/Z_{L/F}, \; \tau \mapsto X\tau^{-1} \) is a bijection.

For every \( U \in U_{N(L/F)}^0 \) there is a unique automorphism \( \tau \in \text{Gal}(L/F) \) satisfying \( U^{1-\varphi} \equiv X\tau^{-1} \mod Z_{L/F}. \)

Thus the map \( N_{L/F}: \text{Gal}(L/F) \rightarrow U_{N(L/F)}^0/Y_{L/F}, \; \tau \mapsto U \)
where \( U \in U_{N(L/F)}^0/Y_{L/F} \) satisfies the equation of the previous paragraph, is a bijection.

**Remark 4.** Thus we get the second reciprocity map \( \mathcal{H}_{L/F}: U_{N(L/F)}^0 \rightarrow \text{Gal}(L/F) \) defined by \( \mathcal{H}_{L/F}(U) = \tau \). The above construction of \( Z_{L/F} \) and \( Y_{L/F} \) provides a new calculation of its kernel.

**Remark 5.** The group \( Z_{L/F} \) for a finite extension \( L/F \) is a subgroup of finite index of \( V(L/F) \).

Recall [10] that \( V(L/F) \) is the connected component of \( \ker N_{L/F} \) in the pro-algebraic Zariski topology. The group \( Z_{L/F} \) is a connected subgroup of finite index of \( V(L/F) \); and one can show that the quotient \( V(L/F)/Z_{L/F} \) has exponent \( \leq p \).

It is a challenging problem to investigate if one can modify the Zariski topology to a new topology \( t \) so that \( Z_{L/F} \) becomes the connected component of \( \ker N_{L/F} \); then one would have a bijection between \( \pi^1_t(U_{L/F})/N_{L/F}\pi^1_t(U_{L}) \) and \( \text{Gal}(L/F) \) similarly to the (geometric) abelian case.

**Remark 6.** It is an open problem if the subgroup \( Z_{L/F} \) depends on the choice of the tower \( E_i \) for nonabelian extensions \( L/F \).
References


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