

MODEL THEORY GUIDANCE IN NUMBER THEORY?

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This short note mentions several areas of number theory and related parts of mathematics where model theory can potentially offer important new insights. Many of the listed above situations are very well known to number theorists. In some of them one can feel important similarities between two mathematical theories, which are still not formalized and well understood. A model theoretical analysis may provide a valuable help.

The main reason to hope for such developments involving model theory, for example as a bridge between two currently separated areas in mathematics, is that in the situations discussed below it is natural to anticipate existence of certain common structures remaining invisible at the current level of knowledge. Model theoretical analysis could help to reveal some of those structures. In some of the situations one seeks a more algebraic construction lying behind analytical objects. And it is well known that model theory does provide a sort of algebraization of analytical constructions.

It is also appropriate to recall that Poizat compares the inclusion model theory – mathematical logic with the inclusion arithmetic – mathematics.

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1. Commutative – noncommutative

The endomorphism ring of a saturated model of a commutative group may become much more noncommutative than the endomorphism ring of the original object. For example, the endomorphism ring of the saturated group of nonstandard integers ${}^*\mathbb{Z}$ is a large noncommutative ring, see [Fe1, sect. 1].

Nonstandard commutative theories may sometimes be related to (parts of) noncommutative theories at the classical level, see [Fe1, sect.4]. This leads to many open questions about nonstandard commutative interpretation or lift of, various objects in representation theory and noncommutative geometry, and its application. See in particular [Fe1, sect. 6–10]. One challenging direction is a development of model theoretical point of view of and its applications to real multiplication programme [Fe1, sect.9], for recent developments see [T].

In particular, [T] introduces the fundamental group of a locally internal space, and it is interesting to study such and other fundamental groups, defined using model theory, from the point of view of their possible relation to motivic fundamental groups.

Generally speaking, objects which are called motivic (fundamental group, cohomology theory, zeta function) seem to be a natural object of analysis for model theorists.

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2. Connecting different characteristics and different p -adic worlds

Many observed but unexplained analogies between theories in characteristic zero and those in positive characteristic, or those in a geometric situation, are well known. The first mathematician who emphasized the importance of such analogies and their use in number theory and algebra was Kronecker.

2a. If one can express various constructions related to the archimedean valuation in a form symmetric to the form for nonarchimedean valuations, this would have many important consequences. See the book [H1] on many aspects of this in relation to zeta functions. First attempts to provide a nonstandard interpretation of some of its ideas are contained in [C].

2b. The anticipated underlying symmetry between primes reveals itself in the Arakelov geometry (see, e.g. [SABK], [Mo]) only to a very partial degree. Model theoretical analysis of its main concepts would be very useful.

2c. It is well known that an analogue of Hurwitz formula in characteristic zero in number field case implies the ABC conjecture, see [Sm]. Could model theory be helpful in getting further insights for such a formula or inequality?

2d. A related activity is the algebra of and algebraic geometry over the "field of one element \mathbb{F}_1 " and also so called absolute derivations, see, e.g. [K1], [K2], [So], [H2], [Dt], [KOW]. They do cry for a model theoretical input.

2e. It was an observation of Tate and Buium that a "non-additive derivative" $\frac{da}{dp}$ with respect to prime p , a in the completion of the maximal unramified extension of \mathbb{Q}_p , could be defined as $\frac{\varphi(a)-a^p}{p}$ where φ is the Frobenius operator. This "non-additive derivative" was used in [Bu]. It would be interesting to have a model theoretical analysis of this "derivative" with respect to p . The work [BMS] provides one of possible answers in this direction.

2f. The expression $\frac{\varphi(a)-a^p}{p}$ is involved in the definition of a p -adic logarithm function and plays a very important role in explicit formulas for the (wild) Hilbert symbol, see [FV, p.259], which themselves are a more elaborate version of much simpler formulas for the tame symbol and formulas for pairings involving differential forms for Riemann surfaces. A model theoretical insight into the unified structure of explicit formulas, which play a fundamental role in arithmetic geometry, would have been of great importance for modern number theory and arithmetic geometry.

3. Representation and deformation theories

3a. The Morita equivalence is not compatible with the standart part map. To which extent this can be used for applications of model theory to representation theory?

3b. Kazhdan's principle in representation theory of reductive groups over local fields says that the theory in positive characteristic zero is often the "limit" of theories in characteristic zero when the ramification index tends to infinity, see e.g. [R]. Model theory in the ramified case is quite difficult, but still it is very interesting to get a model theoretical insight into this observation.

3c. Deformation theory often plays important role, in algebraic geometry, representation theory, group theory, mathematical physics (see e.g. [G], [BGGS], [St]) but we are lacking

a reasonable conceptual understanding of its general structure. Could model theoretical analysis make any progress possible?

4. Higher dimensional objects in arithmetic

Local arithmetic in dimension one and in higher dimensions works with iterated inductive and projective limits of very simple objects, finite abelian p -groups, endowed with several additional structures.

4a. In p -adic representation theory the central role is played by Fontaine's rings (see, e.g. [Be]). They are still waiting for their best invariant and field construction free definition, and a model theoretical analysis may provide it.

4b. Two dimensional local fields, topology on its additive and multiplicative group, their arithmetic, and two dimensional class field theory (see [FK]) are much more difficult than the one dimensional theory. On the other hand, one can view a two dimensional local field as a subquotient of a saturated model of a one dimensional local field (see [Fe2, sect.13]), and hence one can ask for a model theoretical approach to higher local fields and their properties.

4c. Questions on a model theoretical interpretation of translation invariant measure and integration on higher dimensional local fields were asked in [Fe2,Fe3]. The answer has not had to be waited for a long time for: it is contained in the recent work [HK]. In particular [HK] and its further extension unifies the translation invariant integration with the so called motivic integration. This work can be viewed as part of more general systematical use of model theory in algebraic geometry, representation theory and number theory, based on the philosophy of Hrushovski and Kazhdan that model theory allows one to naturally extend the formalism of Grothendieck's approach to the case of algebraic geometry over fields with additional structures, like henselian fields.

5. Quantum physics, and of similarities between it and number theory

5a. Hyperdiscrete (saturation of discrete) constructions in model theory well correspond to the way physicists argue in quantum physics. Combination of both discrete and continuous properties in hyperdiscrete objects is extremely promising for applications in mathematical physics: hyper discrete objects are ideally suited to describe the familiar type of wave-particle behaviour in physics through standard part map images of nonstandard objects.

5b. For first model theoretical insight, from stability point of view, into noncommutative structures of quantum physics see the recent work of Zilber [Z1,Z2].

5c. Divergent integrals ubiquitous in field theories can be naturally viewed via associating to them a nonstandard complex unlimited number, and various renormalization procedures could have enlightening nonstandard interpretations. In particular, nonrigorous physical constructions could be given mathematically sound justification. It is very surprising that almost nothing has been done in this direction. For first steps in nonstandard interpretations of aspects of quantum physic theories see e.g. [Y], [YO].

5d. Much of what is known about quantum field theory comes from perturbation theory and applications of Feynman diagrams for calculation of scattering amplitudes. The Feynman path integral is extremely difficult to give a mathematically sound theory. In

particular, the value of the integral has a rigorous mathematical meaning as a hypercomplex number, manipulations with which do produce standard complex numbers seen in the recipes of Feynman. Recall that Wiener measure (often used in mathematical approaches to the path integral) can be viewed as the Loeb measure associated to hyper random discrete walk. On the other hand, the translation invariant $\mathbb{R}((X))$ -valued measure on two dimensional local fields [Fe2, Fe3] can be viewed as induced from a hyper Haar measure. The additive group of two dimensional local fields, an arithmetic loop space, on which that measure is defined is reasonably close to the loop space on which one calculates the Feynman integral. Could model theory provide a better understanding of the Feynman integral?

5e. Vafa in [Va] indicates that number theory remains the most important part of mathematics which which quantum physics has not had any essential interrelation. Moreover he suggests that quantum mechanics would be reformulated in this century using number theory. I believe that the future understanding of relations of quantum physics with number theory should involve model theory as an interpretor and friendly guide.

Of course many other areas have not been mentioned in this short note, for some of very important (interrelations number theory – complex analysis, number theory – dynamical systems, number theory – algebraic topology) see [Vo], [Dn1], [Dn2], [Ma], [Fu].

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