Reciprocity and IUT

Ivan Fesenko

Galois theory, CFT, IUT

CFT = class field theory, IUT = inter-universal Teichmüller theory = arithmetic deformation theory = Mochizuki theory, 2d = two-dimensional, AAG = adelic analysis and geometry

Diagram of very approximate relations

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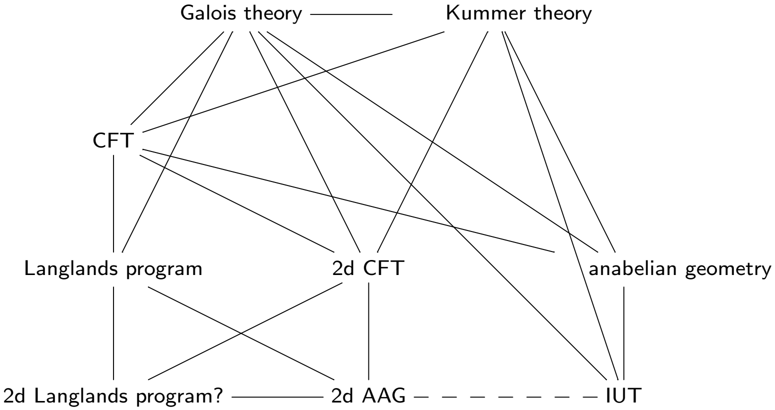

Reciprocity in IUT

IUT uses global data embedded in the product of local data.

Hence at the background there is some categorical anabelian reciprocity.

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Galois groups as tangent bundles

Informally speaking, IUT deals with Galois groups as tangent bundles.

In fact, class field theory does almost the same with abelian Galois groups.

Indeed, abelian Galois groups over a number field k correspond to idele classes,

while adeles are dual to generalised differential forms.

Hence abelian Galois groups are related to the generalised tangent bundle over $\mathrm{Spec}(O_k)$.

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2 components of CFT

CFT mechanism, discovered by Neukirch, can be described on two pages.

Start with an abelian (discrete topological) group A endowed with a continuous action by a profinite group G.

Think of G as the absolute Galois group G_k of a field k.

For an open subgroup G_K of G_k denote by A_K the G_K-fixed elements of A.
Denote by $N_{K/k}\colon A_K \to A_k$ the product of the action of right representatives of G_K in G_k.

Assumption 1: let there be a surjective homomorphism of profinite groups $\deg\colon G_k \to \hat{\mathbb{Z}}$.
Denote its kernel $G_{\tilde{k}}$.

Then for an open subgroup G_K of G_k we get a surjective homomorphism
$\deg_K = |G_k : G_K G_{\tilde{k}}|^{-1} \deg_k\colon G_K \to \hat{\mathbb{Z}}$.

Any element of G_K which is sent by $\deg_K$ to $1 \in \hat{\mathbb{Z}}$ is called a frobenius element w.r.t. $\deg_K$.

Assumption 2: there is a homomorphism $v\colon A_k \to \hat{\mathbb{Z}}$ whose image is $\mathbb{Z}$ or $\hat{\mathbb{Z}}$ and such that $v(N_{K/k} A_K) = |G_k : G_K G_{\tilde{k}}|\, v(A_k)$ for all open subgroups G_K of G_k.

The pair $(\deg, v)$ will define a reciprocity map in the following way.

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Neukirch's CFT mechanism

For a finite extension K of k and a finite Galois extension L/K and σ in its Galois group find any $\tilde{\sigma} \in G(L\tilde{k}/K)$ such that

$\deg(\tilde{\sigma}) \in \mathbb{N}_{\geq 1}$ and $\tilde{\sigma}|_L = \sigma$.

$\mathbb{N}_{\geq 1}$ can be viewed as a frobenius-like object inside étale-like object $\hat{\mathbb{Z}}$, from the point of view of IUT.

Denote by Σ the fixed field of $\tilde{\sigma}$. Then $L\tilde{k}/\Sigma$ is an unramified extension w.r.t. v and $\tilde{\sigma}$ is a frobenius element of G_Σ.

Call $\pi_K \in A_K$ such that $|\hat{\mathbb{Z}} : \deg(G_K)|^{-1} v(N_{K/k}(\pi_K)) = 1$ a prime element of A_K.
Then π_K remains prime in all finite subextensions of $K\tilde{k}/K$, so the latter is unramified w.r.t. v.

Define the reciprocity map

$$\Psi_{L/K} \colon \sigma \mapsto N_{\Sigma/K}\pi_\Sigma \mod N_{L/K}A_L,$$

where π_Σ is a prime element of A_Σ.

If appropriate axioms for A under the action of G (axioms of CFT) are satisfied, then

- $\Psi_{L/K}$ is well defined,
- $\Psi_{L/K}$ induces an isomorphism $G(L/K)^{ab} \to A_K/N_{L/K}A_L$,
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For a local field of mixed characteristic one can take the $\hat{\mathbb{Z}}$-extension as the maximal unramified extension generated by roots of order prime to the residue characteristic, but one can also consider other (ramified) $\hat{\mathbb{Z}}$-extensions.

Usually one uses the canonical frobenius automorphism to fix deg and takes v as the usual surjective discrete valuation, then one gets the canonical local reciprocity map.
However, the theory works for an arbitrary deg and v satisfying the assumptions.
In particular, there is an indeterminacy in local class field theory.

For a number field k the only $\hat{\mathbb{Z}}$-extension of $\mathbb{Q}$ is the unique $\hat{\mathbb{Z}}$-subextension of $\mathbb{Q}$ inside its maximal abelian extension $\mathbb{Q}^{ab}$ generated by all roots of unity.

The map v is first defined on ideles as the composite of the product of the local reciprocity maps (with the canonical local frobenius automorphisms) restricted to $G(\tilde{k}/k)$ and then composed with deg.

The reciprocity law for cyclotomic extensions of $\mathbb{Q}$ implies that v factorises through idele classes.

There is no canonical choice of a generator of $G(\tilde{\mathbb{Q}}/\mathbb{Q})$, hence no canonical choice of deg, but the global reciprocity map does not depend on this choice, since using $a\,\deg$ with $a \in \hat{\mathbb{Z}}^{\times}$ changes v to av and the pairs $(\deg, v)$ and $(a\deg, av)$ define the same reciprocity map.

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This CFT mechanism is purely group theoretical and does not depend on ring structures.

This makes it non-alien to the IUT structures.

However, to verify the CFT axioms for local or global fields one has to use ring structures.

Dozens of papers on class formations aimed to derive CFT from as little as possible.

From the point of view of IUT in distinguishing monoidal structures from ring structures, the long term search for class formations has the meaning of distinguishing purely monoid theoretical aspects of CFT (CFT mechanism) from its ring theoretical aspects (proving axioms of CFT).

CFT mechanism is similar to Kummer theory mechanism in the sense that both are purely monoid theoretical.

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Problem 1. **Find possible applications of the CFT mechanism in IUT**.

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p-CFT of cdvf with perfect residue field of char p,

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1d CFT (including relatively poorly known theories, not a full list)

CFT of 1d global fields, 1d local fields, i.e. complete discrete valuation fields (cdvf) with finite residue field,

p-CFT of cdvf with perfect residue field of char p,

p-CFT of cdvf with imperfect residue field of char p (unusual: no CFT induction on degree)

nonabelian CFT, with 1-cocycle reciprocity map, of arithmetically profinite Galois extensions of local fields,

with further contributions by Ikeda and Serbest, recently extended to global fields by Ikeda

1d CFT and radiality

Computations in Galois cohomologies and of the Brauer group can be used for CFT but are not needed for CFT.

Thus, the knowledge of the Brauer groups is not intrinsic to CFT.

IUT uses computation of the Brauer group of a local fields for cyclotomic rigidity algorithms for local fields.

These algorithms are of uniradial type, not of multiradial type which is the preferred type of algorithms for IUT.

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2d CFT

2d CFT (first expositions originally developed in Japan by K. Kato and Sh. Saito) uses Milnor K_2-groups or even better, their quotients $K_2^t = K_2/\cap_{m\geq 1} mK_2$

◇ 2d global fields K – function fields of integral normal 2d schemes S flat over $\mathbb{Z}$ or $\mathbb{F}_p$, including integral curves over 1d global fields;

◇ 2d local fields $K_{x,y}$, $x \in y \subset S$, e.g. $\mathbb{Q}_p((t))$, $\mathbb{R}((t))$, $\mathbb{C}((t))$, $\mathbb{Q}_p\{\{t\}\}$;

◇ 2d local-global fields, e.g. the function field K_y of the completion of the local ring of a curve $y \subset S$ (cdvf with global residue field),

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Note: unlike 1d CFT, 2d CFT is separated from various geometrical issues, e.g. CFT for surfaces does not know about the Picard groups of the surfaces, see below.

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Functoriality, multiradiality, Kummer theory, evaluation at points and CFT

Functoriality properties in CFT are related to Galois automorphism action, finite separable extensions, passing to residue fields, passing to completions.
More elementary in comparison to CFT Kummer theory is compatible with Galois evaluation, evaluation of functions at torsion points.

From the point of view of abelian theories such as CFT, multiradiality issues are not important. They may be important for nonabelian CFT.

On the other hand, multiradiality is of fundamental importance in IUT.
The use of Kummer theory and evaluation of functions at torsion points is crucial for multiradial issues.

Unlike special types of CFT, such as cyclotomic and CM, which use evaluation of functions at torsion points, general CFT follows different conceptual patterns.

Currently it is not known whether general CFT is compatible with evaluation of functions on curves over 1d fields at their points of the type used in IUT.

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To some extent, aspects of IUT involving Galois evaluation, evaluation of functions on curves, related to an elliptic curve, over number fields at points of the curves, can be viewed as a global realisation of a generalisation of Kronecker's dream at the level of Kummer theory.

Its local realisation is the Lubin–Tate theory.

Problem 2. **Find a version of (nonabelian) CFT which is compatible with evaluation of functions on curves over 1d fields at their points, of the type used in IUT. Check its place in relation to Kronecker's Jugendtraum**.

κ-coric rational functions on hyperbolic curves and their values at certain special points play a key role in IUT.

Problem 3. **Develop further the concept and theory of κ-coric functions and explore its use in CFT**.

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CFT and anabelian geometry

An argument in anabelian geometry sometimes uses a reduction to the case of abelian Galois extensions.

Early work in anabelian geometry used CFT (or closely related theories) in

- 1d theory for global fields (Neukirch, Ikeda, Uchida),
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CFT and IUT

IUT uses the computation of the Brauer group of a local field but not local CFT and not global CFT.

Product formula in IUT is very different from the product formula construction in CFT: global reciprocity law depends on nontrivial relationships between local units at one place and elements of local value groups at another place of the number field. This is incompatible with the (canonical) splittings of local units and local value groups playing a key role for the theta link in IUT. Hence product aspects of in IUT, which use 2d structures (the fundamental group of a curve over a number fields), differ very much from those in CFT.

Problem 4. **Investigate generalisations of the categorical definitions of global part of theatres in IUT and its product formula construction from the point of view of a new type of general adelic objects in CFT (1d and 2d) and their use.**

Other concepts of IUT such as mono-anabelian algorithms, monoid theoretic considerations, categorical geometry aspects, synchronizations, indeterminacies may stimulate some further extensions and developments of CFT (1d and 2d).

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2d structures in IUT and 2d CFT

Let k be a local or global field of characteristic zero. Let C be a hyperbolic (integral normal) curve over k. Let K be the function field of C. Let G_K be the absolute Galois group of K.

From the profinite group G_K

one can go to

(a) $\pi_1(C)$, the Galois group over K of the maximal subextension K_C of a maximal separable extension of K, the compositum of finite Galois subextensions of K s.t. the corresponding morphism of proper curves is étale over C and each finite Galois subextension of K_C/K comes from a curve étale over C.

A theorem of Mochizuki recovers G_K from $\pi_1(C)$ for certain hyperbolic orbi-curves C.

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Both $\pi_1(C)$ and G_K^{ab} are two-dimensional.

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2d CFT deals with bad reduction fibres of proper regular elliptic surfaces by working with appropriate K-delic objects: the quotient of the restricted product $\prod'_{x\in y} K_2^t(K_{x,y})$ of two-dimensional local fields $K_{x,y}$ (of type $\mathbb{Q}_p\{\{t\}\}$) modulo the image of $K_2(K_y)$, K_y is the fraction field of the completion of the local ring of the fibre y.

Problem 5. **Find a 2d CFT proof of the main results of ET.**

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Two 2d adelic structures on surfaces and two symmetries of IUT

The two symmetries in IUT:

geometric additive $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry and arithmetic multiplicative $\mathbb{F}_\ell^{\circledast}$-symmetry

have a number of features which are reminiscent of

geometric additive 2d adelic structure, related to rank 1 integral structure. Its duality is more powerful than Serre's duality and it implies the Riemann–Roch theorem on srufaces,

analytic/arithmetic multiplicative 2d adelic structure, related to rank 2 integral structure. It underlies 2d zeta integral of surfaces, related to the zeta- and L-functions and it is non-scheme-theoretical.

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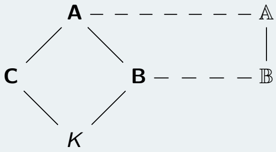
Six adelic objects ($ABCK$) on surfaces S:



Objects on the left hand side are geometric adeles:

A double restricted product of local-local $K_{x,y}$,

B restricted product of local-global K_y for curves y,

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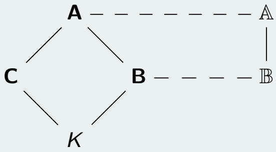

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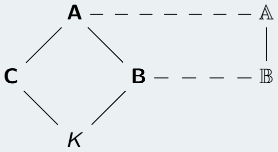

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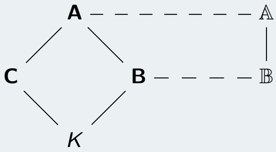

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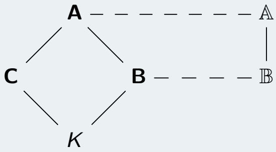

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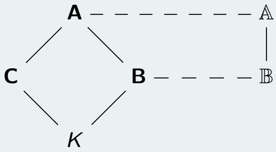
Two 2d adelic structures on surfaces and two symmetries of IUT

Six adelic objects $(ABCK)$ on surfaces S:



Objects on the left hand side are geometric adeles:

A double restricted product of local-local $K_{x,y}$,

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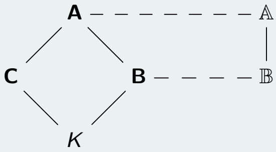

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$$K_2^t(\mathbf{A})/(K_2^t(\mathbf{B}) + K_2^t(\mathbf{C})) \to G_K^{\mathrm{ab}}$$

2d CFT does not involve relations between objects at the bottom two levels, such as

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Zeta integral and IUT

Various analogies between aspects of IUT and the computation of the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ have already been discussed at the workshop.

On the other hand, there are various analogies between the computation of the Gaussian integral and two computations of

the zeta integral $\zeta(f,s)$, both 1d and 2d.

In particular,

the use of cartesian coordinates for two copies of the Gaussian integral corresponds to the use of the restricted product definition of ideles

$$\mathbb{A}^\times = \prod{}' k_v^\times$$

in the definition of the zeta integral and its and zeta function factorization into the product of the Euler factors (in the terminology of IUT, frobenius-like structure),

the use of spherical coordinates corresponds to the use of the second term of the filtration on ideles

$$\mathbb{A}^\times \supset \mathbb{A}_1^\times \supset k^\times,$$

where $\mathbb{A}_1^\times$ are ideles of module 1.
This filtration is most natural from CFT point of view and hence from étale-like (in the terminology of IUT) point of view.
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The nonarchimedean theta-function

$$\theta(u) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)/2} u^n = (1-u) \prod_{n\geq 1} \left((1-q^n)(1-q^n u)(1-q^n u^{-1})\right)$$

in IUT is closely related to the complex theta-function

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

The function $\theta(0, ix) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 x)$ in real variable x can be written as

$$\sum_{x\in\mathbb{Q}} f(x) = \int_{\mathbb{Q}} f \, d\mu,$$

where $f = \otimes\, \mathrm{char}_{\mathbb{Z}_p}(x) \otimes \exp(-x^2/2)$ is an eigenfunction of the adelic unitary Fourier transform with eigenvalue 1.

The multiplicative Fourier transform (i.e. the Mellin transform) of $\theta(0, ix) - 1$

$$\int_0^\infty (\theta(0, ix) - 1) x^s dx/x = \int_{\mathbb{A}^\times} f(x)|x|^s d\mu_{\mathbb{A}^\times}(x) = \zeta(f, s)$$

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2d AAG and IUT

Generalisations of these properties play a crucial role in two-dimensional adelic analysis and geometry on elliptic surfaces.

Two-dimensional zeta integral for a proper model of an elliptic curve over a global field involves, instead of $\mathrm{char}_{\mathbb{Z}_p}$, the tensor product of characteristic functions of integral rings of rank 2 of 2d local fields as components of f

$$\zeta(f,s) = \int_{\mathbb{A}^\times \times \mathbb{A}^\times} f(x)\, ||x||^s \, d\mu(x).$$

This zeta integral is equal to the square of the zeta function of the model times auxiliary factors.

In particular, the zeta integral computation includes a two-dimensional adelic formula for the norm of the minimal discriminant and conductor of the elliptic curve.

Possible interaction of 2d AAG and IUT may lead not only to further applications of IUT but also to applications of 2d AAG to the generalized Szpiro conjecture.

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Bridges between arithmetic and geometry

Bridges between arithmetic and geometry are very natural and relatively easy in IUT, and they are used in the construction of theatres.

Bridges between arithmetic and geometry in 2d AAG are non-trivial. Various two-dimensional objects have two integral structures, of rank 1 and of rank 2. At the level of additive groups, these structures are very different.

The study of 2d zeta integral uses an interplay between the multiplicative groups of the two adelic structures on surfaces via the symbol map and structures important for 2d CFT

$$\begin{array}{ccccc} \mathbb{A}^\times \otimes \mathbf{A}^\times & & & & \\ \downarrow & \searrow & & & \\ \mathbb{A}^\times \times \mathbb{A}^\times & \longrightarrow & K_2^t(\mathbf{A})/(K_2^t(\mathbf{B}) + K_2^t(\mathbf{C})) & \longrightarrow & G_K^{ab}. \end{array}$$

Geometric (multiradial) containers for arithmetic (radial) data are of fundamental importance in IUT.
In 2d AAG, a kind of analog of such geometric container is $\mathbb{A}^\times \otimes \mathbf{A}^\times$ in the previous diagram.

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Using the computation of 2d zeta integral, 2d AAG interprets the BSD conjecture for elliptic curves and elliptic surfaces as the consequence of a bridge between the two adelic structures on the surfaces and as a bridge between geometry and arithmetic.

Since bridges between geometry and arithmetic in IUT are natural and play a fundamental role, it is time to set the following

Problem 6. **Investigage interaction of IUT and 2d AAG and possible applications to the BSD conjecture**.

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