ABOUT CERTAIN ASPECTS OF THE STUDY AND DISSEMINATION OF SHINICHI MOCHIZUKI’S IUT THEORY

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This text aims to communicate in a compact form some of factual information related to the math study of Sh. Mochizuki’s IUT theory1 and its dissemination, as well as various aspects of the situation around IUT such as its perception by non-experts. Some of more general issues are discussed in two other papers2,3. Without repeating the content of those papers, this text deals with some concrete issues, rather related to some mathematicians than to mathematics. The summary of what follows is this.

The main prerequisite for IUT is the vast area of (arithmetic) anabelian geometry developed since 1990 in Japan. In 2012 there were few experts in arithmetic anabelian geometry outside Japan. IUT is not an increase of mathematical knowledge in an area in which there are many specialists able to study it. It is a rare pioneering vast development with many new concepts and ideas, and with a great potential for future developments. To become an expert in IUT, one has to invest an adequate large amount of time in a dedicated serious focused study of the theory starting with its prerequisites. This cannot be done in the period of few weeks or months. To help mathematicians to study IUT, a very large amount of time and effort has been dedicated to the dissemination of IUT, via various workshops, including large international, via seminars, lectures and study groups. No valid math evidence of any serious fault in IUT has been found by anyone. Minor oversights have been found and corrected. To this day there remains no mathematically substantive reason whatsoever to doubt the validity of IUT. The number of researchers who have mastered IUT, by investing a large amount of time and effort, is steadily growing and is a two-digit one. These researchers have sent a 4-digit number of questions and remarks to the author, all addressed. 11 text-surveys of IUT and a book on IUT by 9 mathematicians from 5 countries individually present the theory in different ways. There are more mathematicians able to produce expert reports on the IUT papers than the number of such reports on previous rare major breakthrough math results at the time of their publication. In 2018–2019 two year long IUT seminars at RIMS for new learners were conducted. 2021 is a special RIMS year with 4 international workshops on anabelian geometry, combinatorial anabelian geometry and IUT.

Some have tried to study IUT on the own, but have not been able to proceed far, this is normal. There are many younger researchers who in the course of several years of hard study of IUT, have asked interesting questions and contributed to new original developments. Unusually, but reflecting the period of time we live in, there were few researchers, lacking any expertise in anabelian geometry, who were active in publicly making at best ignorant and at worst dishonest remarks about IUT, always devoid of any valid math evidence of faults in the theory.4 There were few people who spread disinformation about IUT that might have affected some

1 The IUT papers made public in August 2012 are available from section Inter-universal Teichmüller Theory of its author page http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html. See those pages for various information on seminars and workshops on IUT. See also this page https://www.maths.nottingham.ac.uk/plp/pmzibf/guidestoIUT.html.
3 I. Fesenko, Remarks on aspects of modern pioneering mathematical research, available from https://www.maths.nottingham.ac.uk/plp/pmzibf/ripp.pdf
4 compare with ‘we are also fighting on a second front that we did not anticipate, fighting a battle against misinformation and disinformation in a hyper-partisan environment ... the world has changed in profound ways since even 2010. Social media, hyper-partisanshhip, the broad populist distrust of experts, plummeting standards of factfulness’, https://twitter.com/CT_Bergstrom/status/1243252341756669953
mathematicians in other areas, unable to distinguish an expert in the relevant subject area from a non-expert, and non-mathematicians. This kind of unprofessional behaviour should be strongly rejected. Its roots lie deep and potentially harm the future of math; fixing the issue requires a serious determined attitude.

1. On mathematical environment around IUT, briefly. Class field theory, the heart of algebraic number theory, has several important generalisations. They include the Langlands correspondences, anabelian geometry and higher class field theory. By various reasons the first generalisation has attracted many times more researchers than the second and the third, but all of these generalisations of class field theory are fundamentally important. Most of the central problems in the second and third generalisations of class field theory have been settled. One can imagine another universe where higher class field theory and anabelian geometry attract many more researchers than in this universe and where general class field theory concepts are well understood and used for fundamental achievements in the Langlands program. In this universe the main conjectures over number fields in the Langlands program remain open, despite the well known achievements in some very special cases and fundamental advances in the functional and geometric cases.

The main prerequisite for IUT theory of Sh. Mochizuki is arithmetic anabelian geometry, including his famous proofs of the Grothendieck conjecture and his mono-anabelian geometry. Arithmetic anabelian geometry was started in works of Neukirch–Ikeda–Uchida–Iwasawa for small fields (such as number fields or their completions) in characteristic zero, and from a different motivation for hyperbolic curves over number fields it was proposed by Grothendieck. The main leading country in arithmetic anabelian geometry is Japan, and the first three contributors to anabelian geometry were H. Nakamura, A. Tamagawa and Sh. Mochizuki. Below ‘anabelian geometry’ will mean ‘arithmetic anabelian geometry’. In the last thirty year a vast body of fundamentally important results in anabelian geometry were established. These developments were essentially left unnoticed outside a small group of experts.

The IUT theory uses some key theorems in anabelian geometry, as well as its later developments such as absolute anabelian geometry and mono-anabelian geometry. The total volume of relevant papers in anabelian geometry used in one or another extent in IUT is huge, even though it is possible not to read all of its 1500 pages. One starting observation for arithmetic deformation theory, i.e. IUT, is that unlike the usual algebraic geometry in which working with schemes corresponds to working with rings, working with certain anabelian objects corresponds to working with topological groups, thus using one operation instead of two, with new options to perform kinds of arithmetic deformation, not available in the standard arithmetic geometry that is locally commutative ring theory. There is an associated difficulty to measure the deviation of certain diagrams of groups and maps between groups from being commutative, which IUT solves, thus giving bounds on deformation.

2. The study of IUT. Links to various study materials about IUT are available from pages of the author of IUT. The total amount of time dedicated to the verification process of IUT by mathematicians is several decades, and it looks to be the largest time ever spent in the history of mathematics on the verification of a mathematical work prior to its publication. Several international conferences were organised in 2014-2016. Numerous intensive seminars have been held in Japan (2012-2019), UK (2015-2018), China (2015-2016), they involved nationals of many countries. In addition to the referees’ comments, active learners of IUT sent a 4-digit number of comments, questions, remarks, all had been carefully taken into account by the author. Several learners of IUT

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5 For more details and related math issues see https://www.maths.nottingham.ac.uk/plp/pmzibf/232.pdf.
6 In the 1990s, a series of results about anabelian properties of Galois groups of global and higher global fields, i.e. birational anabelian geometry, were obtained by F. Pop. Since the early 1990s, F. Bogomolov suggested and developed, later in collaboration with Yu. Tschinkel, his birational anabelian geometry for varieties of dimension $> 1$ over algebraically closed fields, this theory is quite different from arithmetic anabelian geometry in many respects.
7 http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html
shared their understanding of it, by having written texts and surveys. There are already more surveys of IUT than of any previous fundamental work at the time of its publication.⁸

Small numbers of experts in anabelian geometry, overall poor digestion of Grothendieck’s 50 years old math heritage, a relatively large distance between anabelian geometry and IUT and the mainstream directions, a large number of new concepts in IUT and its relatively large volume, as well the general current situation with top fundamental research in number theory have substantially affected the study IUT. Some people applied serious efforts to study IUT for some time, but stopped—indeed, the task is huge. IUT can be a difficult theory to study for experts from other standard number theoretical areas, since they cannot easily apply their previous expertise without learning first, as PhD students, new for them areas such as arithmetic anabelian geometry. It is crucial to appreciate that whatever are one’s previous results in other areas of number theory, they do not make one an expert in anabelian geometry and IUT.

Recommendations to mathematicians interested to study IUT. Pathways to study IUT are available from many sources including www-links in footnotes of this text. If you find a piece of IUT looking to you as an error, and you cannot resolve it, document your evidence and contact the author or experts to discuss.

3. On negative aspects of reaction to IUT.

3.1. On reaction to IUT from some mathematicians. Of course, many mathematicians are interested to know more about IUT, but, as always in mathematics, one should follow experts opinions. It is still possible to contribute useful questions, comments, remarks, e.g. in relation to more conventional parts of the theory, e.g. those that came in 2012 from Dimitrov and Venkatesh.

More researchers in the midst of their career than ever prefer to stay within their narrow area and show no enthusiasm or ability to learn new groundbreaking theories. Mathematicians who can understand conventional things quickly, who can master a domain that has already been well established may have not good chances to progress in the study of IUT.⁹ Several expected candidates to study the theory chose to be as unambitious as it can get by doing essentially nothing about its study for seven years. Some preferred to adopt the convenient stance of sceptical attitude not based on any expert knowledge of the subject area.

A naïve request to provide more details and explanations for the IUT papers and a psychologically comfortable intention to wait for this to happen demonstrates sheer lack of basic knowledge of the situation: the level of presentation of IUT was very detailed in 2012 and in the last 7 years the volume of the IUT papers had increased by some 100 pages more, reaching almost 600 pages, there are already 11 surveys of IUT and even a book for the general audience.

Who can doubt that any professional consensus about any mathematical theory can only come from experts in its subject area. Mochizuki’s work includes fundamental contributions in numerous directions: Hodge–Arakelov theory, anabelian geometry, mono-anabelian geometry, combinatorial anabelian geometry, Grothendieck–Teichmüller group, p-adic Teichmüller theory, inter-universal Teichmüller theory. Except for the last direction, none of his work has ever been criticised — because it was read and appreciated by experts. Critical comments are potentially useful when they come from experts. Rarely for mathematical developments, some mathematicians have publicly said ignorant negative things about IUT and its study while being fully aware they do not have any expertise in the subject area. In the first approximation, the number of unjustifiable negative reactions to IUT, not based on valid math knowledge, was inversely proportional to the number of home academicians capable to study the theory. There was only expert in arithmetic anabelian geometry in the USA in 2012, but most of incorrect negative comments originate from a tiny group of US mathematicians. Some chose to spread a malicious distortion of the math truth or false rumours¹⁰.

⁸ For their incomplete list see e.g. http://www.maths.nottingham.ac.uk/plp/pmzibf/guidestoiut.html.
⁹ https://www.bbc.co.uk/news/world-europe-50856999
¹⁰ See e.g. the report about the Oxford IUT workshop https://www.maths.nottingham.ac.uk/plp/pmzibf/files/iut-i-rep.html
One of them is talking about some kind of controversy about the status of IUT — however, to have a controversy about a mathematical work there should be genuine experts on both sides of the argument able to provide valid math arguments, but this is not the case for IUT. There is no proven study or expertise in anabelian geometry and IUT among grumblers and no valid math evidence to justify their position. This explains why not serious math peer review but shallow posting is the only venue for their chat about IUT.

One can meet an uninformed opinion among people far away from anabelian geometry that there are allegedly two mathematical sides in their take on IUT—this is factually wrong: there is only one side, the side of experts in IUT, which includes many those who have worked for years to learn the subject area and the theory. They, together with the referees and the group of editors processing the IUT papers, have all concluded that the IUT papers have no mathematical flaws. Part of this process was a truly unprecedented event with the author of IUT kept investing a lot of time in answering more than 1000 of questions for more than 7 years.11

Something is fundamentally rotten here, and it has to be addressed properly. Why do some mathematicians make public their opinions about a fundamental development in the subject area where they have empty research record, with no evidence of their serious study of it, and without providing any math evidence of errors in the theory? If one is reckless, one should learn the error of making hasty judgments and come to appreciate the difference between the superficial and the essential. The less innocent reasons are pursuing goals having nothing to do with the theory they chose to say negative things about.

3.2. Articles about IUT in mass media of some countries. IUT has attracted a high level of interest from mass media. There are some reasonably good written articles about IUT and its author. At the same time, there are irresponsibly written articles presenting very inaccurate pictures.

Most experts on IUT decline to answer journalists questions, so then journalists contact mathematicians or even laypersons with zero publication record in number theory. Some of the interviewed mathematicians are good in their own areas, but that does not make them experts in areas they do not know; unfortunately, the articles including their interviews fail to make clear the level of their non-expertise. Experience in areas such as classical Diophantine geometry, algebraic geometry, modularity, Galois representations or aspects of p-adic geometry does not enable one with the expert intuition and knowledge of anabelian geometry and IUT.

One of easiest ways for journalists to write their articles is to present opposite points of view but in the case of IUT the journalists often fail to appreciate that they mix experts opinions (all of which are positive) with ignorant opinions of non-specialists who are not kept in the loop in relation to the study of IUT. It is similar to as if an article about the value of a graduate course is written by mixing opinions of its students with top exam grade and its students who never attended any lecture and failed the exam.

Recommendation to serious journalists. Before interviewing a mathematician about IUT, first check the expertise level by asking several simple questions such as their knowledge of and expertise in anabelian geometry, talks on anabelian geometry at international conferences, the number of hours spent on the study of IUT and whether they asked questions about IUT to the author of IUT or experts in IUT.

3.3. One of unsuccessful attempts to study IUT. In 2013–2017 not a single concrete mathematical remark indicating a serious problem in IUT was produced. Since 2014 one mathematician P. Scholze, no publications or expertise in anabelian geometry, kept talking publicly about faults in IUT without ever providing any math evidence.12\textsuperscript{13} By and by, after a lot of pressure, Scholze visited RIMS, together with J. Stix, in March 2018, just

\textsuperscript{11}compare this intensive study and verification with the next section material
\textsuperscript{12}The author of this text t wrote to Scholze several times asking to behave professionally and in particular to tell precisely what were the faults in IUT he knew about. Eventually, he sent just one most loosely stated question to Mochizuki in May 2015. The author of IUT responded to him with a long email that also offered to conduct discussions via email to address any questions, but Scholze, not surprisingly, declined to communicate further. Part of this is stated on p.3 of the main Mochizuki’s report, see footnote 13.
for 5 days. After the meeting which revealed almost zero level of understanding of IUT by the German mathematicians, they came with their perfidious caricature version of IUT based on a lazy person oversimplification of IUT in which they identify all isomorphic rings and ‘forget’ about the fundamental role of automorphism groups in anabelian geometry. Their dull report essentially denies the use of anabelian geometry and infinitely many theatres in IUT\textsuperscript{14}. The grasp of the IUT theory in it is below that of a 2nd year PhD student studying anabelian geometry and IUT. Initially, Scholze and Stix intended to put their report about the meeting online. However, after reading Mochizuki’s report on their report, see especially its sect. 17-18\textsuperscript{15} and these comments\textsuperscript{16}, they changed their mind in July 2018 and stopped to be interested to post their own report but agreed to let the author of IUT to include their reports on his pages. In his extensive report on their report the author of IUT formulated several questions to the German mathematicians which may have helped them to appreciate how erroneous was their take on IUT.\textsuperscript{17} The second version of their report did not address most of comments of Mochizuki on their first report. It also included new incorrect statements in classical knowledge such as a blunder in height theory and a fundamental misunderstanding of one of the Faltings work, which can be easily seen by many. The wrong take on IUT by the two German mathematicians, who are even unable to answer very few questions asked to them by the author of IUT, is rejected by all experts and can not pass any careful peer review process\textsuperscript{18} but is nevertheless perceived by some non-experts and ignorant journalists as ‘another side’ in the argument about IUT.\textsuperscript{19} That short lived study of IUT stands in shark contrast with the deep study of it by the other mathematicians mentioned above, who asked/made many good questions, remarks and comments. The failure of those two German mathematicians in their rushed study of IUT should not and it does not stop serious researchers to study IUT.

4. Developments. There are already several new math developments related to IUT, in different directions.

Four international workshops on anabelian geometry and IUT are organised during a special RIMS Project Research year on Expanding Horizons of Inter-universal Teichmüller Theory in 2021\textsuperscript{20}, supported by the new Center for Research in Next-Generation Geometry.

A book\textsuperscript{21} by F. Kato, published in April 2019, presents various features of IUT to the wider audience. This book was in the list of top twenty bestselling books in all subject areas on amazon in Japan, and it was awarded the Yaesu prize\textsuperscript{22}.

5. On April 3 2020 at the press-conference of Kyoto University, ran by M. Kashiwara and A. Tamagawa, it was announced that the IUT papers are accepted for publications and will soon be published.\textsuperscript{23}