Sequential topologies and quotients of Milnor K-groups of higher local fields

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Abstract. For a higher local field F algebraic properties of $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ of the Milnor K-group are studied by using topological and arithmetical considerations. In particular, standartness of the torsion in the latter group and divisibility of the group $\cap_{l\geqslant 1} lK_m(F)$ if the last residue field of F is finite are proved. It is shown that $\cap_{l\geqslant 1} lK_m(F)$ coincides with the intersection of all neighbourhoods of zero in $K_m(F)$ (and if F is an m-dimensional local field, it equals to the kernel of the reciprocity map $K_m(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$). A description of $K_m(F)/l$ in the language of topological generators and relations is provided. The equality of several topologies on $K_m(F)$ at the level of subgroups is shown and their difference is discussed.

Appendix of this paper written by O. Izhboldin describes a construction of a field F containing a primitive pth root and such that p-torsion of $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ is not generated by p-torsion in F^*. He works with the field of rational functions of an infinite product of certain Severi–Brauer varieties.

1. Introduction

An n-dimensional local field $F = k_n$ is a complete discrete valuation field with residue field k_{n-1} being $(n-1)$-dimensional; 0-dimensional local field is a perfect field k_0 of positive characteristic p. Thus to every n-dimensional field F corresponds $n+1$ fields $k_n = F, k_{n-1}, \dots, k_0$.

Lifting prime elements of $k_n, \dots, k_1$ to the field F one obtains an ordered system of local parameters $t_n, \dots, t_1$ (t_n is a prime element of F).

Let $s \geqslant 0$ be the minimal integer such that $\mathrm{char}(F) = \mathrm{char}(k_s)$. If $s < n$, then F is isomorphic to $k_s((t_{s+1}))\dots((t_n))$. Note that the group of principal units of F with respect to the discrete valuation of rank $n-s$ is divisible if $\mathrm{char}(F) = 0$, and so it is not very interesting from the point of view of class field theory. The field k_s if $s \neq 0$ is called a mixed characteristic field, it is a natural higher dimensional analogue of a $\mathfrak{p}$-adic field.

From many points of view Milnor K-groups of F are not the most suitable object for a meaningful description of abelian extensions of an n-dimensional local field F in higher local class field theory; the structure of Milnor K-groups of F is still not completely known. It is more convenient and natural to work with quotients $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ of Milnor K-groups endowed with a special topology. Arithmetical homomorphisms from Milnor K-groups (like a reciprocity map) factorize through such quotients.

In this paper we discuss algebraic properties of $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ by using topological and arithmetical considerations which are related to topological and arithmetical properties of higher local fields. In turn, even though applications are not discussed here, the properties of the quotients

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of the K-groups are quite important for the arithmetic of higher local fields due to higher local class field theory.

We provide a new short approach which replaces longer approaches in [2–5]. It corrects and clarifies some statements or proofs of [15], [2–5], [21] and links between them.

For the reader convenience the text contains almost all definitions. New methods and results of this paper are:

(1) simultaneous work with several topologies,

(2) correction of Parshin's theorem on description of the subgroup of topological Milnor K-groups in positive characteristic case,

(3) new proofs in the case of finite residue field k_0 of the following results:

(a) $\cap_{l\geqslant 1} lK_m(F)$ coincides with the intersection $\Lambda_m(F)$ of all open subgroups in $K_m(F)$ with respect to the topologies on $K_m(F)$ mentioned above.

(b) $\cap_{l\geqslant 1} lK_m(F)$ is a divisible group.

(c) If a primitive rth root of unity is contained in F then r-torsion of

$$K_m^{\mathrm{top}}(F) = K_m(F)/\cap_{l\geqslant 1} lK_m(F)$$

is generated by r-torsion in F^*.

Section 2 presents several topologies on the additive and multiplicative group of F which are defined by induction on n. They are different from Parshin's topology [15] in characteristic p. Their common feature is that each of them has the same set of convergence sequences. The multiplication is sequentially continuous but not necessarily continuous with respect to some of them. For class field theory sequential continuity seems to be more important than continuity. This is a hidden phenomenon in dimension 1 and 2, where continuity is the same as sequential continuity. In particular, this can affect our understanding of a generalization of harmonic analysis to higher multidimensional fields.

Three important pairings of Milnor K-groups are described in section 3. They are in intensive use in section 4 when studying properties of Milnor K-groups of F endowed with topologies. We compare various topologies (λ_m in 4.1, ν_m, o_m, σ_m in 4.6) on Milnor K-groups some of which were in use in [2–5] and show that they all coincide at the level of subgroups. In subsections 4.5 – 4.7, central in this paper, we prove (a)–(c). In fact we prove more general results in the case of a perfect k_0; for details see 4.5 – 4.7.

Concerning the structure of $K_m^{\mathrm{top}}(F)$: it is completely known in characteristic p (most of results except (a) and (b) were stated by Parshin in [15]) and we supply its (partial) description in characteristic 0. As an application in subsection 4.8 we deduce the standard description of the kernel of the norm map for K_m^{top} (which is not currently established for K_m of an arbitrary field if $m > 2$).

The results of this paper are important for explicit higher local class field theory as presented in [2–5]; there are further applications, for instance to explicit formulas in class field theory, study of abelian extensions, Fukaya's map for Milnor K_2-groups of complete discrete valuation fields with residue field having one element p-basis.

Appendix of this paper written by late O. Izhboldin describes a construction of a field F containing a primitive pth root and such that p-torsion of $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ is not generated by p-torsion in F^*. He works with the field of rational functions of an infinite product of certain Severi–Brauer varieties.

Quite a different construction for irregular prime numbers p follows from works of G. Banaszak [24]. First, it is easy to see that if p-torsion of $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ is generated by p-torsion in F^*, then $\cap_{l\geqslant 1} lK_m(F) = p\cap_{l\geqslant 1} lK_m(F)$ and hence the group $\cap_{l\geqslant 1} lK_m(F)$ is either zero or infinite.

Now using [23, p.289] for each irregular prime number p there is a totally real field K (maximal totally real subfield of $\mathbb{Q}(\mu_p)$) such that the p-primary part of the group $\cap_{l\geqslant 1} lK_2(K)$ is non-zero. Let $F = K(\mu_p)$. Then the p-primary part of the finite [22, Th.8.9] group $\cap_{l\geqslant 1} lK_2(F)$ is non-zero and therefore p-torsion of $K_2(F)/\cap_{l\geqslant 1} lK_2(F)$ is not generated by p-torsion in F^*. I am grateful to G. Banaszak for correspondence on this subject.

Throughout the text we denote by μ_m the group of mth roots of unity. For an abelian group A we denote m-torsion points of A by $\operatorname{Tors}_m A$.

2. Topology on the multiplicative group

By $\mathcal{O}_F$ we denote the ring of integers of F with respect to the discrete valuation of rank n associated to $t_n, \ldots, t_1$; $\mathcal{O}_F$ doesn't depend on the choice of a system of local parameters. Denote by $V_F = 1 + (t_n, \ldots, t_1)\mathcal{O}_F = 1 + t_1\mathcal{O}_F$ the group of principal units of F as an n-dimensional local field.

Denote by $\mathcal{O}_0$ the subring in F corresponding to the last residue field k_0 if $\operatorname{char}(F) = p$ and the image in F of the ring of Witt vectors of k_0 if $\operatorname{char}(F) = 0$. The ring $\mathcal{O}_0$ contains the set of multiplicative representatives $\mathcal{R}$ of k_0.

2.1. As an example, for a 2-dimensional local field F with a system of local parameters t_2, t_1 define a base of neighborhoods of 1 as $1 + t_2^i\mathcal{O}_F + t_1^j\mathcal{O}_0[[t_1, t_2]]$ (e.g. [7]). Then every element $\alpha \in F^*$ can be expanded as a convergent with respect to the just defined topology product

$$\alpha = t_2^{a_2} t_1^{a_1}\theta \prod (1 + \theta_{i,j} t_2^i t_1^j)$$

with $\theta \in \mathcal{R}^*$, $\theta_{i,j} \in \mathcal{R}$, $a_i \in \mathbb{Z}$. The set $S = \{(j, i) : \theta_{i,j} \neq 0\}$ satisfies the property: for every i there is j_i such that $(j, i) \in S$ implies $j \geqslant j_i$. Call such a set admissible.

2.2. We provide a short descrition of topology λ on the additive group of F in the case of $\operatorname{char}(k_{n-1}) = p$. For a more detailed discussion see [26].

Definition. The topology λ on k_0 is discrete.

Fix a system of local parameters t_i. The residue field k_{n-1} can be identified with the field $k_0((\overline{t_1}))\ldots((\overline{t_{n-1}}))$. Let F_0 be a complete discrete valuation subfield of F which contains the elements $t_{n-1}, \ldots, t_1$ and a prime element $t = t_n$ if F is of characteristic p and $t = p$ if F is of characteristic 0, and which has the residue field $k_0(\overline{t_1})\ldots(\overline{t_{n-1}})$. Fix a lifting (and thus a set of representatives S) of $k_0(\overline{t_1})\ldots(\overline{t_{n-1}})$ in F_0 such that elements of k_0 are mapped to their multiplicative representatives in $\mathcal{O}_0$, and the residues $\bar{t}_i \in k_{n-1}$, $1 \leqslant i \leqslant n-1$, are mapped to t_i in F_0. By linearity this determines the lifting of $k_0(\overline{t_1})\ldots(\overline{t_{n-1}})$ to F_0.

Given the topology λ on the additive group k_{n-1}, introduce the topology λ on the additive group F.

First, an element $\alpha \in F_0$ is said to be a limit of a sequence of elements $\alpha_v \in F_0$ if and only if given any series $\alpha_v = \sum_i \theta_{v,i} t^i$, $\alpha = \sum_i \theta_i t^i$ with $\theta_* \in S$, for every set $\{U_i, \quad -\infty < i < +\infty\}$ of neighbourhoods of zero in k_{n-1} and every i_0 for almost all v the residue of $\theta_{v,i} - \theta_i$ belongs to U_i for all $i < i_0$. Second, a subset U in F_0 is called open if and only if for every $\alpha \in U$ and every sequence $\alpha_v \in F_0$ having α as a limit almost all α_v belong to U. This determines the topology λ on F_0.

Now one proves by induction on n that the completion of F_0 with respect to λ is a subfield E of F. The field F is in fact a finite dimensional vector space over E and thus F is endowed with the topology λ. This topology doesn't depend on the choice of a system of local parameters [14].

It is not difficult to deduce the following properties (for some relevant details see [14]):

(1) α is a limit of α_v if and only if the sequence α_v converges to α with respect to the topology λ;

(2) a limit is uniquely determined;

(3) each Cauchy sequence with respect to the topology λ converges in F;

(4) the limit of the sum of two convergent sequences is the sum of their limits;

Remark. Let F be of characteristic p. The topology λ on the additive group is different from that introduced by Parshin in [15] for $n \geqslant 2$: for example, the set $W = F \setminus \{t_2^a t_1^{-c} + t_2^{-a} t_1^c : a, c \geqslant 1\}$ in $F = \mathbb{F}_p((t_1))((t_2))$ is open in the just defined topology, ie for each convergent sequence $x_v \to x \in W$ almost all x_v belong to W. If for some open subgroups U_i in the additive group of $\mathbb{F}_p((t_1))$ such that $U_i = \mathbb{F}_p((t_1))$ for $i \geqslant a$ the group $\{x = \sum a_i t_2^i : x \in F, a_i \in U_i\}$ were contained in W, then for any positive c such that $t_1^c \in U_{-a}$ we would have $t_2^a t_1^{-c} + t_2^{-a} t_1^c \in W$, a contradiction.

However, a sequence of elements in F converges to $x \in F$ with respect to λ if and only if it converges with respect to the topology introduced by Parshin. In fact, the topology λ is the finest topology in which the set of convergent sequences is the same as in the topology introduced by Parshin. So λ can be viewed as the sequential saturation of the Parshin topology.

2.3. Definitions. If char(k_{n-1}) = p, then define a topology λ on F^* as the product of the induced from F topology on the group of principal units V_F, the discrete topologies on the cyclic groups generated by t_i and the discrete topology on $\mathcal{R}^*$.

If char(F) = char(k_s) = 0, char(k_{s-1}) = p, then define a topology λ on F^* as the product of the trivial topology on $1 + (t_n, \dots, t_{s+1})\mathcal{O}_F$ (which is a divisible subgroup of F^*), the discrete topology on the cyclic groups generated by t_i with $i > s$ and the topology λ on k_s^*.

2.4. The following properties of the topology λ on F^* can be deduced by induction on dimension n.

Properties.

(1) Each Cauchy sequence with respect to the topology λ converges in F^*, the limit of the product of two convergent sequences is the product of their limits. The multiplication in F^* is sequentially continuous.

(2) For a 2-dimensional local field its multiplicative group F^* is a topological group and it has a countable base of open subgroups (for example see [7]). In the case of a F with $n \geqslant 3$ and $s \geqslant 2$ both assertions don't hold. For example, let F be a two-dimensional field and let $L = F((t_3))$. If $Z = 1 + Wt_3 + t_3^2 F[[t_3]]$ is an open subgroup in $1 + t_3 F[[t_3]]$, $W \subset F$, then W is an open subgroup in F. It is easy to see that then $WW = F$. So the coefficient of t_3^2 in ZZ can be any element of F. Plenty of open subgroups Y of $1 + t_3 F[[t_3]]$ don't satisfy the property that the coefficient of t_3^2 in Y can be any element of F. Therefore for those open subgroups Y there is no open subgroup Z of L such that $ZZ \subset Y$.

(3) For every open neighbourhood U of 1 in F^* there is r such that $V_F^{p^r} \subset U$ (see [21, Lemma 1.6]).

(4) For any sequence $a_i \in V_F$ the sequence $a_i^{p^i}$ converges to 1. If $\mathcal{M} = (t_n, \dots, t_l)\mathcal{O}_F$ then any sequence $a_i \in 1 + \mathcal{M}^i$ converges to 1.

(5) Subgroups $V_F^{p^r}$ are closed in V_F. The product of $V_F^{p^r}$ and a closed subgroup in V_F is closed.

REMARK. If $\operatorname{char}(F) = p$ and k_0 is finite then the topology λ on the multiplicative group is different from that introduced by Parshin in [15]. For example, for $n \geqslant 3$ each open subgroup A in F^* with respect to the topology introduced in [15] possesses the property: $1 + t_n^2 \mathcal{O}_F \subset (1 + t_n^3 \mathcal{O}_F)A$ (indeed, it is easy to see that the product of two open subgroups of k_{n-1} is equal to k_{n-1}; this implies the indicated property). However, the subgroup in $1 + t_n \mathcal{O}_F$ topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$ with $(i_n, \dots, i_1) \neq (2, 1, \dots, 0)$ (all zeros except the first two components), $i_n \geqslant 1$ (ie the sequential closure of the subgroup generated by these elements), is open in λ and doesn't satisfy the above-mentioned property. So the topology λ and the topology of [15] are distinct even at the level of subgroups. Another, third topology on F^* is discussed in subsection 2.6.

2.5. DEFINITION. Call a subset X of elements of $(\mathbb{Z})^n$ greater than 0 admissible if for every $1 < m \leqslant n$ and every $(i_m, \dots, i_n)$ there is $j(i_m, \dots, i_n)$ such that $(i_1, \dots, i_n) \in X$ implies $i_{m-1} \geqslant j(i_m, \dots, i_n)$, and there is j such that all $(i_1, \dots, i_n) \in X$ implies $i_n \geqslant j$.

If $\operatorname{char}(k_{n-1}) = 0$ then every element $\alpha \in F^*$ is a product of an infinitely divisible element, a power of t_n and an element of k_{n-1}^*.

If $\operatorname{char}(k_{n-1}) = p$ then every element $\alpha \in F^*$ is a convergent product

$$\alpha = t_n^{a_n} \dots t_1^{a_1} \theta \prod (1 + \theta_{i_n,\dots,i_1} t_n^{i_n} \dots t_1^{i_1}), \qquad \theta \in \mathcal{R}^*, \theta_{i_n,\dots,i_1} \in \mathcal{R}$$

with $\{(i_1, \dots, i_n) : \theta_{i_n,\dots,i_1} \neq 0\}$ being an admissible set (see [4], [14]).

2.6. Include a principal unit $\varepsilon \in V_F$ into an arbitrary topological basis $\{\varepsilon_\alpha\}$ of V_F and consider the subgroup topologically generated by $t_n^{a_n}, \dots, t_1^{a_1}$, $\mathcal{R}^*$, principal units $\varepsilon_\alpha \notin \langle \varepsilon \rangle$ and ε^{p^s}. It is an open subgroup of finite index of F^*. Denote the shift-invariant topology ν on F^* which has these open subgroups as the base of neighbourhoods of 1. A sequence of elements in F^* converges to 1 with respect to λ if and only if it converges with respect to ν. The group F^* is a topological group with respect to ν.

Since for every subgroup H of F^* and an element $\alpha \in F^*$ such that there is an open subset U of F^* which includes H and doesn't contain α there is an open subgroup V which includes H and doesn't contain α, every subgroup of F^* which is the intersection of some open subsets is the intersection of some open subgroups. In particular, the intersection of all open subgroups of finite index containing a closed subgroup H coincides with H. Hence for any subgroup H of F^* a sequence of elements of F^*/H converges with respect to the quotient topology of ν if and only if it converges with respect to the quotient topology of λ. Thus, if two sequences converge in F^*/H, then their product converges to the product of their limits.

For other definitions and details see [7] ($n = 2$), [15], [2–5], [14].

3. Pairings of K-groups of higher local fields

Let $G_{\mathrm{ur,ab},p}$ be the Galois group of the maximal abelian unramified p-extension of F over F (which corresponds to the maximal p-extension of the last residue field k_0).

3.1. Let F be an n-dimensional local field of characteristic p. For $\alpha_1, \dots, \alpha_n \in F^*$, a Witt vector $(\beta_0, \dots, \beta_r) \in W_r(F)$ and $\varphi \in G_{\mathrm{ur,ab},p}$ put

$$(\alpha_1, \dots, \alpha_n, (\beta_0, \dots, \beta_r)]_r(\varphi) = (\varphi - 1)(\gamma_0, \dots, \gamma_r)$$

where $(\mathrm{Frob}-1)(\gamma_0, \ldots, \gamma_r) = (\lambda_0, \ldots, \lambda_r)$; and the ith ghost component $\lambda^{(i)}$ of $(\lambda_0, \ldots, \lambda_r)$ is $\mathrm{res}_{k_0}(\beta^{(i)}\alpha_1^{-1}d\alpha_1 \wedge \cdots \wedge \alpha_n^{-1}d\alpha_n)$. In fact, to make the previous definition precise one needs, as usual with Witt vectors, to pass to Witt vectors over a ring of characteristic zero, like $\mathbb{Z}_p((t_1))\ldots((t_n))$ and then return back.

This is a sequentially continuous and symbolic (ie satisfies the Steinberg property) in the first n coordinates map. It defines the *Artin–Schreier–Witt–Parshin pairing* [15] (finite k_0 case), [4]:

$$K_n(F)/p^r \times W_r(F)/(\mathrm{Frob}-1)W_r(F) \to \mathrm{Hom}\left(G_{\mathrm{ur,ab},p}, W_r(\mathbb{F}_p)\right)$$

where Frob is the Frobenius map.

3.2. Let F be an n-dimensional local field of characteristic 0 and let a primitive p^rth root of unity ζ be contained in F. Suppose that p is odd (if $p = 2$ then formulas become much more complicated).

Suppose first that $\mathrm{char}(k_{n-1}) = p$.

Let $X_1, \ldots, X_n$ be independent indeterminates over the quotient field of $\mathcal{O}_0$ (the latter is defined at the beginning of section 2). For an element

$$\alpha = t_n^{a_n} \ldots t_1^{a_1}\theta \prod (1 + \theta_{i_n,\ldots,i_1} t_n^{i_n} \ldots t_1^{i_1})$$

of F^*, with $\theta \in \mathcal{R}^*$, $\theta_{i_n,\ldots,i_1} \in \mathcal{R}$ put

$$\alpha(X) = X_n^{a_n} \ldots X_1^{a_1}\theta \prod (1 + \theta_{i_n,\ldots,i_1} X_n^{i_n} \ldots X_1^{i_1}).$$

The formal power series $\alpha(X) \in \mathcal{O}_0((X_1))\ldots((X_n))$ depends on the choice of local parameters and the choice of a power series expression of α. Denote $z(X) = \zeta(X)$, $s(X) = z(X)^{p^r} - 1$. Define the action of the operator Δ on θ's and on X_i as raising to the pth power. For $\alpha \in F^*$ put $l(\alpha) = p^{-1} \log \alpha(X)^{p-\Delta}$.

Now for elements $\alpha_1, \ldots, \alpha_{n+1} \in F^*$ define $\Phi(\alpha_1, \ldots, \alpha_{n+1})$ as

$$\sum_{i=1}^{n+1} (-1)^{n+1-i} l(\alpha_i) \left(\frac{\mathrm{d}\,\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\,\alpha_{i-1}}{\alpha_{i-1}} \wedge p^{-1}\frac{\mathrm{d}\,\alpha_{i+1}^{\Delta}}{\alpha_{i+1}^{\Delta}} \wedge \cdots \wedge p^{-1}\frac{\mathrm{d}\,\alpha_{n+1}^{\Delta}}{\alpha_{n+1}^{\Delta}} \right).$$

Define the Vostokov map [19] (case of finite k_0), [4]:

$$V_r\colon (F^*)^{\oplus n+1} \to \mathrm{Hom}(G_{\mathrm{ur,ab},p}, \mu_{p^r})$$

as

$$V_r(\alpha_1, \ldots, \alpha_{n+1})(\varphi) = \zeta^{(\varphi-1)\delta}, \qquad \text{where } (\mathrm{Frob}-1)\delta = \mathrm{res}\,\Phi(\alpha_1, \ldots, \alpha_{n+1})/s(X)$$

for $\varphi \in G_{\mathrm{ur,ab},p}$.

This map V_r doesn't depend on the attaching formal power series to elements of F and the choice of a system of local parameters. The map V_r is sequentially continuous and symbolic, so it induces a homomorphism

$$V_r\colon K_{n+1}(F)/p^r \to \mathrm{Hom}(G_{\mathrm{ur,ab},p}, \mu_{p^r}).$$

The latter is in fact an isomorphism, as follows from the results of section 4.7.

For $\chi \in \mathrm{Hom}(G_{\mathrm{ur,ab},p}, \mu_{p^r})$ denote by $E(\chi)$ any principal unit such that $V_r(\{t_1, \ldots, t_n, E(\chi)\} = \chi$. The elements $E(\chi)$ in their explicit form introduced by Vostokov as primary elements in [19, sect. 1] in the case of finite k_0 play an important role in deducing the explicit formula.

If $\mathrm{char}(F) = \mathrm{char}(k_s) \neq \mathrm{char}(k_{s-1})$ then define the *Vostokov pairing*

$$K_m(F)/p^r \times K_{n+1-m}(F)/p^r \to \mathrm{Hom}(G_{\mathrm{ur,ab},p}, \mu_{p^r})$$

using the canonical surjections $K_1(F)/p^r \to K_1(k_s)/p^r$.

3.3. The composition of border homomorphisms in K-theory (eg, [9, Ch.IX, sect. 2]) supplies a homomorphism

$$\partial_m: K_m(F) \to K_m(k_0) \oplus \cdots \oplus K_0(k_0).$$

The kernel of ∂_n is the subgroup $VK_n(F)$ defined in 4.1.

We also get pairings $K_m(F) \times K_{n+1-m}(F) \to K_{n+1}(F) \xrightarrow{\partial_{n+1}} K_m(k_0) \oplus \cdots \oplus K_0(k_0)$.

In particular, the tame symbol $\mathfrak{c}$ is the composition of ∂_{n+1} with the projection to $K_1(k_0)$ and the lifting $K_1(k_0) \to \mathcal{R}^*$ can be explicitly described as follows. For an element $\alpha \in F^*$ and its expression as in 3.2 put $v^{(j)}(\alpha) = a_j$ for $1 \leqslant j \leqslant n$. For elements $\alpha_1, \ldots, \alpha_{n+1}$ of F^* the value $\mathfrak{c}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ is the element of $\mathcal{R}^*$ whose residue is equal to the residue of $\alpha_1^{b_1} \ldots \alpha_{n+1}^{b_{n+1}}(-1)^b$ in the last residue field k_0, where $b = \sum_{s,i<j} v^{(s)}(\alpha_i)v^{(s)}(\alpha_j)b_{i,j}^s$ and b_j is the determinant of the matrix obtained by omitting the jth column with the sign $(-1)^{j-1}$ from the matrix $A = (v^{(i)}(\alpha_j))$, and $b_{i,j}^s$ is the determinant of the matrix obtained by omitting the ith and jth columns and sth row from A.

4. K^{top}-groups

4.1. Definition. Let λ_m be the finest topology on $K_m(F)$ for which the map

$$\phi: F^{*\oplus m} \to K_m(F), \quad \phi(\alpha_1, \ldots, \alpha_m) = \{\alpha_1, \ldots, \alpha_m\}$$

is sequentially continuous with respect to the product of the topology λ on F^* (section 2) and for which the subtraction in $K_m(F)$ is sequentially continuous. Define

$$K_m^{\text{top}}(F) = K_m(F)/\Lambda_m(F)$$

with the quotient topology where $\Lambda_m(F)$ is the intersection of all neighbourhoods of 0 with respect to λ_m (and so is a subgroup).

Remark 1. Every sequentially open (closed) subset with respect to λ_m is open (resp. closed). The topology λ_m coincides with the finest topology on $K_m(F)$ for which the map ϕ is sequentially continuous with respect to the product of the topology ν on F^* (defined in 2.6) or Parshin's topology [15] and for which the subtraction in $K_m(F)$ is sequentially continuous.

From the definition it follows that λ_m is a shift invariant topology. We have $\lambda = \lambda_1$ and $K_1^{\text{top}}(F)$ coincides with the quotient of F^* by the maximal divisible subgroup of V_F. If the first residue field k_{n-1} is of characteristic p and the last residue field k_0 is finite, then $K_1^{\text{top}}(F) = K_1(F)$.

Definition. Denote by $VK_m(F)$ the subgroup of $K_m(F)$ generated by V_F; similarly introduce $VK_m^{\text{top}}(F)$.

Remark 2. The induced topology on $VK_m(F)$ by λ_m coincides with the the finest topology on $VK_m(F)$ for which the restriction of ϕ on $V_F \oplus F^{*\oplus m-1} \to VK_m(F)$ is sequentially continuous with respect to the product of the topology λ on F^* and for which the subtraction in $VK_m(F)$ is sequentially continuous. Indeed, the map ϕ and subtraction in $K_m(F)$ is sequentially continuous with respect to the shift invariant topology on $K_m(F)$ extended from the finest topology on $VK_m(F)$ as above.

A sequentially continuous symbol homomorphism from the tensor product of m copies of F^* to a topological Hausdorff group G induces a continuous homomorphism from $K_m^{\text{top}}(F)$ to G. Therefore the Artin–Schreier–Parshin, Vostokov pairings and tame symbol defined in section 3 are factorized through topological K-groups.

From property (5) of 2.4 one deduces that $\{\theta, \varepsilon\} = 0$ in $K_2^{\text{top}}(F)$ for $\theta \in \mathcal{R}^*$ and a principal unit ε.

Suppose that k_0 is algebraic over $\mathbb{F}_p$. Then

(1) $\{\theta, \theta'\} = 0$ in $K_2(F)$ for $\theta, \theta' \in \mathcal{R}^*$. More generally, if the absolute Galois group of k_0 is procyclic then $\{\theta, \theta'\} \in \cap_{l \geqslant 1} l K_2(F)$ for $\theta, \theta' \in \mathcal{R}^*$ (this is due to the fact that the norm map for a finite extension of k_0 is surjective).

(2) Using the tame symbol defined in 3.3 one easily shows that $K_m(F)$ splits into the direct sum of $VK_m(F)$, several copies of $\mathbb{Z}$ and of the group $\mathcal{R}^*$.

(3) $K_m^{\text{top}}(F)$ splits into the direct sum of $VK_m^{\text{top}}(F)$, several copies of $\mathbb{Z}$ and of the group $\mathcal{R}^*$. In particular, $K_n^{\text{top}}(F)$ is isomorphic to the direct sum of the cyclic group generated by $\{t_1, \ldots, t_n\}$, n copies of $\mathcal{R}^*$, and the subgroup $VK_n^{\text{top}}(F)$.

REMARK 3. In the general case of a perfect residue field k_0 the group $K_m^{\text{top}}(F)/p$ splits into the direct sum of $VK_m^{\text{top}}(F)/p$ and several copies of $\mathbb{Z}/p$.

4.2. For two principal units $\varepsilon, \eta \in F^*$ in $K_2(F)$ the following formula is straightforward:

$$\{\varepsilon, \eta\} = \{1 - \varepsilon, 1 + (\varepsilon - 1)\eta\} - \{1 + (\varepsilon - 1)(\eta - 1)\varepsilon^{-1}, \eta\}.$$

The principal unit $1+(\varepsilon-1)(\eta-1)\varepsilon^{-1}$ is of higher order than that of ε, η. Now for $\beta \in (t_n, \ldots, t_1)\mathcal{O}_F$ and $\theta \in \mathcal{R}^*$, we get

$$\begin{aligned}\{1 - \theta t_n^{i_n} \ldots t_1^{i_1}, 1 + \beta\} &= \{\theta t_n^{i_n} \ldots t_1^{i_1}, 1 - \theta t_n^{i_n} \ldots t_1^{i_1}(1 + \beta)\} \\ &\quad - \{1 - \theta t_n^{i_n} \ldots t_1^{i_1}\beta(1 - \theta t_n^{i_n} \ldots t_1^{i_1})^{-1}, 1 + \beta\}.\end{aligned}$$

The first symbol of the right hand side can be written as the sum of symbols $\{t_i, \lambda_i\}$ with principal units λ_i which sequentially continuously depend on $1 - \theta t_n^{i_n} \ldots t_1^{i_1}$ and $1 + \beta$, and the first element of the second symbol is closer to 1 than $1 - \theta t_n^{i_n} \ldots t_1^{i_1}$ is.

For an arbitrary principal unit ε factorize it into the convergent product of $1 - \theta t_n^{i_n} \ldots t_1^{i_1}$ and then apply the previous argument. One can verify following Parshin's method [15, sect.2] (put $\eta = 1 + \beta$) that the symbol $\{\varepsilon, \eta\}$ can be written in $K_2^{\text{top}}(F)$ as $\sum_i \{\rho_i, t_i\} + \{\rho, \eta\}$ with principal units ρ_i, ρ which sequentially continuously depend on ε and η and ρ closer to 1 than ε is. This implies (repeating the argument) that for arbitrary principal units ε, η the symbol $\{\varepsilon, \eta\}$ can be written in $K_2^{\text{top}}(F)$ as $\sum \{\rho_i, t_i\}$ with principal units ρ_i which sequentially continuously depend on ε and η.

Therefore, every element x of $VK_m(F)$ can be written as a sum of an element of $\Lambda_m(F)$ plus a fixed number of elements of the form $\{\alpha_i\} \cdot \{\text{some local parameters}\}$ with $\alpha_i \in F^*$.

REMARK. Since lx for $x \in VK_m(F)$ can be written as the sum of symbols $\{\alpha^l\} \cdot \{\text{local parameters}\}$ ($\alpha \in V_F$) and an element of $\Lambda_m(F)$, from property (4) of 2.4 we get

$$\cap_r p^r VK_m^{\text{top}}(F) = \{0\}.$$

4.3. From the previous subsection we immediately deduce that the group $VK_m^{\mathrm{top}}(F)$ is topologically generated (with admissible sets playing the same role as in the case of F^*) by symbols $\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_{j_1}, \dots, t_{j_{m-1}}\}$ with $\theta \in \mathcal{R}$.

Topological relations among these generators (modulo p^r for each r in the case of char(F) = p, modulo p^r in the case char(F) = 0 and a primitive p^rth root of unity belongs to F, modulo p if char(F) = 0 and a primitive pth root of unity doesn't belong to F) are revealed using the Artin–Schreier–Witt–Parshin and Vostokov pairings, for details see [15], [2–5]. Simultaneously one verifies that appropriately modified pairings are nondegenerate.

For example, if $\mu_{p^r} \subset F^*$, then the Vostokov pairing V_r is very useful in the study of the structure of $K_m^{\mathrm{top}}(F)/p^r$ in terms of topological generators and relations between them, see [2, sect. 3 and 4] (the quotient filtration on $K_m(F)/p$ induced by the standard filtration on F^* as a discrete valuation field can be also described in terms of differential forms over k_{n-1} by Bloch–Kato's result [1]).

Using explicit calculations with the Vostokov pairing one shows that $p^r K_m^{\mathrm{top}}(F)$ coincides with the intersection of open subgroups of $K_m^{\mathrm{top}}(F)$ containing $p^r K_m^{\mathrm{top}}(F)$. One can also prove the following lemma (the method of the proof is entirely similar to that of [2]).

LEMMA. *Let F be an n-dimensional local field containing a primitive pth root. Let $L = F(\mu_{p^r})$ and $p^s = |L : F|$. Let σ be a generator of* Gal(L/F). *Then the annihilator of $i_{F/L} K_{n+1-m}^{\mathrm{top}}(F)$ with respect to the Vostokov pairing V_r is equal to $(\sigma - 1)K_m^{\mathrm{top}}(L) + p^{r-s} i_{F/L} K_m^{\mathrm{top}}(F) + p^r K_m^{\mathrm{top}}(L)$.*

4.4. DEFINITION. For topological spaces A_i, $1 \leqslant i \leqslant d$, introduce the following $*$-product topology on $\oplus A_i$: first, an element $(x_1, \dots, x_d) \in \oplus A_i$ is a limit of a sequence $(x_{v,1}, \dots, x_{v,d}) \in \oplus A_i$ if $x_{v,i} \to x_i$ when $v \to +\infty$ for all $1 \leqslant i \leqslant d$. Now a subset $Y \in \oplus A_i$ is called an open subset if for every $y \in Y$ and every sequence $y_v \in \oplus A_i$ having y as a limit almost all y_v belong to Y.

The $*$-product topology is in general strictly finer than the topology of the product of topological spaces; it is the sequential saturation of the product topology. For example, if $n = 3$, char(F) = p, then the set $F^* \times F^* \setminus \{(1 + t_3 t_2^a t_1^{-c}, 1 + t_3 t_2^{-a} t_1^c) : a, c \geqslant 1\}$ is open in $F^* \times F^*$ with respect to the $*$-product topology, but is not open in the product topology.

Compare this definition with the definitions in 2.2.

4.5. In subsections 4.5–4.7 we study relations between $\Lambda_m(F) \cap VK_m(F)$ and $\cap_{l \geqslant 1} lVK_m(F)$ in the general case of a perfect k_0.

REMARK. If k_0 is algebraic over $\mathbb{F}_p$ then $\Lambda_m(F) \subset VK_m(F)$ and $\cap_{l \geqslant 1} lK_m(F) = \cap_{l \geqslant 1} lVK_m(F)$ as follows from 4.1. In general $\cap_{l \geqslant 1} lK_m(F) \not\subset VK_m(F)$ (because the divisible part of $\mathcal{R}^*$ can be nontrivial; see also (1) in 4.1).

THEOREM. *If* char(F) = p *then $\Lambda_m(F) \cap VK_m(F)$ coincides with $\cap_{l \geqslant 1} lVK_m(F)$; the group $\Lambda_m(F) \cap VK_m(F)$ is p-divisible.*

Proof. According to Bloch–Kato–Gabber theorem [1] the differential symbol

$$d_F \colon K_m(F)/p \to \Omega_F^m, \qquad d_F\{\alpha_1, \dots, \alpha_m\} = \frac{\mathrm{d}\,\alpha_1}{\alpha_1} \wedge \dots \wedge \frac{\mathrm{d}\,\alpha_m}{\alpha_m}$$

is injective.

Ω_F^m is a finite-dimensional vector space over F/F^p, so the intersection of all neighborhoods of zero in Ω_F^m with respect to the induced by λ topology is trivial. The differential symbol d_F is continuous. Therefore its injectivity implies $\Lambda_m(F) \subset pK_m(F)$.

Let J consist of $j_1, \dots, j_{m-1}$ and run all $(m-1)$-elements subsets of $\{1, \dots, n\}$, $m \leqslant n+1$. Let $\mathcal{E}_J$ be the subgroup of V_F generated by $1+\theta t_n^{i_n} \cdots t_1^{i_1}$, $\theta \in \mathcal{O}_0$, with restrictions that p doesn't divide $\gcd(i_1, \dots, i_n)$ and the smallest index l for which i_l is prime to p doesn't belong to J. Consider the induced by λ topology on $\mathcal{E}_J$. Applications of the Artin–Schreier–Witt–Parshin pairing provide a proof of a corrected theorem of Parshin (the proof goes in the same way as in [15]) which claims that there exists an isomorphism and homeomorphism ψ from the group $\prod_J \mathcal{E}_J$ (with the $*$-product of the induced topology by Λ on $\mathcal{E}_J$) onto $VK_m^{\mathrm{top}}(F)$. This provides an explicit and satisfactory description of the topology on $K_m^{\mathrm{top}}(F)$ in the positive characteristic case.

Therefore the group $VK_m(F)/\Lambda_m(F) \cap VK_m(F) = \psi(\prod \mathcal{E}_J)$ doesn't have nontrivial p-torsion. The group $K_m(F)/VK_m(F)$ has no nontrivial p-torsion due to 3.3 (since $\mathrm{Tors}_p\, K_i(k_0) = \{0\}$). Now from $\Lambda_m(F) \subset pK_m(F)$ we deduce that $\Lambda_m(F) \cap VK_m(F) = p(\Lambda_m(F) \cap VK_m(F))$.

Since V_F is l-divisible for every l prime to p, we get

$$\Lambda_m(F) \cap VK_m(F) \subset \cap_{l \geqslant 1} lVK_m(F).$$

For the inverse inclusion see Remark in 4.2. $\square$

4.6. For $m \leqslant n+1$, $d = \binom{n}{m-1}$ define the homomorphism

$$g\colon V_F^{\oplus d} \to VK_m(F), \quad \beta_J \mapsto \sum \{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\},$$

where $J = \{j_1, \dots, j_{m-1}\}$ runs over all $m-1$ elements subsets of $\{1, \dots, n\}$.

THEOREM.

(i) $\mathrm{im}(g) + \Lambda_m(F) \cap VK_m(F) = VK_m(F)$. *The homomorphism*

$$g_0\colon V_F^{\oplus d}/g^{-1}(\Lambda_m(F)) \to VK_m^{\mathrm{top}}(F)$$

is a homeomorphism between $V_F^{\oplus d}/g^{-1}(\Lambda_m(F))$ *with the quotient topology of the* $*$*-product topology of the induced by* λ *topology on* V_F *and* $VK_m^{\mathrm{top}}(F)$ *with the topology* λ_m.

(ii) $\Lambda_m(F) \cap VK_m(F)$ *is the intersection of all open subgroups of finite index in* $VK_m(F)$.

Proof. From the definitions and 4.2 it follows that for $\alpha_1 \in V_F$, $\alpha_2, \dots, \alpha_m \in F^*$ there exist elements $\beta_J \in V_F$, $J = \{j_1, \dots, j_{m-1}\}$, which sequentially continuously depend on $\alpha_1, \dots, \alpha_m$ such that the symbol $\{\alpha_1, \dots, \alpha_m\}$ can be written as

$$\sum \{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\} \mod \Lambda_m(F).$$

So there is a sequentially continuous map $f\colon V_F \times F^{*\oplus m-1} \to V_F^{\oplus d}$ such that its composition with g coincides with the restriction of the map ϕ on $V_F \oplus F^{*\oplus m-1}$ modulo $\Lambda_m(F)$.

Let U be an open subset in $VK_m(F)$. Then $g^{-1}(U)$ is open in the $*$-product of the topology λ on $V_F^{\oplus d}$. Indeed, otherwise for some J there were a sequence $\alpha_J^{(i)} \notin g^{-1}(U)$ which converges to $\alpha_J \in g^{-1}(U)$. Then the properties of the map ϕ imply that the sequence $\phi(\alpha_J^{(i)}) \notin U$ converges to $\phi(\alpha_J) \in U$ which contradicts the openness of U. Thus, $g^{-1}(\Lambda_m(F))$ is the intersection of open subsets of $V_F^{\oplus d}$.

The product of two convergent sequences x_i, y_i in $V_F^{\oplus d}/g^{-1}(\Lambda_m(F))$ converges to the product of their limits by 2.6. Hence using Remark 2 of 4.1 we deduce that the quotient topology of λ_m on $VK_m(F)/\Lambda_m(F) \cap VK_m(F)$ is $\geqslant$ the quotient topology of the $*$-product of λ on $V_F^{\oplus d}$ via g.

Thus,

$$g\colon V_F^{\oplus d}/g^{-1}(\Lambda_m(F)) \to VK_m(F)/\Lambda_m(F) \cap VK_m(F)$$

is a homeomorphism of $V_F^{\oplus d}/g^{-1}(\Lambda_m(F))$ with the quotient of the $*$-product topology of the λ topologies on $V_F^{\oplus d}$ and of $VK_m(F)/\Lambda_m(F)\cap VK_m(F)$ with the quotient topology of λ_m.

Note that $\Lambda_m(F)$ is closed: if $x_i \in \Lambda_m(F)$ tends to x, then $x = x_i + y_i$ where x_i, y_i converges to 0, so x converges to 0 and hence belongs to $\Lambda_m(F)$. To deduce (ii) use the fact that every closed subgroup in $V_F^{\oplus d}$ is the intersection of certain open subgroups of finite index (see 2.6), hence every closed subgroup in $VK_m(F)$ is the intersection of certain open subgroups of finite index. □

REMARK 1. Using property (4) of 2.4 and the previous theorem we deduce that for sets of subgroups $\{Z_i\} = \{p^i VK_m^{\mathrm{top}}(F)\}$ or $\{Z_i\} = \{\langle\{1+\mathcal{M}^i\}K_{m-1}^{\mathrm{top}}(F)\rangle\}$ where $\mathcal{M} = (t_n, \dots, t_l)\mathcal{O}_F$ then the natural map

$$VK_m^{\mathrm{top}}(F) \to \varprojlim VK_m^{\mathrm{top}}(F)/Z_i$$

is surjective.

REMARK 2. Introduce on $VK_m^{\mathrm{top}}(F)$ the topology ν_m induced by the topology ν on V_F via the map g. Due to the properties of ν in 2.6 the group $VK_m^{\mathrm{top}}(F)$ is a topological group with respect to ν_m and by property (ii) of the preceding theorem the intersection of all neighbourhoods of zero in $VK_m^{\mathrm{top}}(F)$ with respect to λ_m is zero. Since ν and λ have the same sets of convergent sequences, ν_m and λ_m have also the same sets of convergent sequences.

REMARK 3. From the proof of the theorem we deduce that λ_m coincides with the finest topology o_m on $K_m(F)$ such that the map

$$(F^{*\oplus m})^{\oplus d} \to K_m(F), \qquad (\beta_{j,i}) \mapsto \sum_{1\leqslant i\leqslant d} \{\beta_{1,i}, \dots, \beta_{m,i}\}$$

is sequentially continuous.

REMARK 4. From the proof of the theorem we deduce that λ_m coincides with the finest topology σ_m on $K_m(F)$ such that for all $\alpha_i \in F^*$, $1 \leqslant i \leqslant m-1$, the map from F^* to $K_m(F)$, $\alpha \to \{\alpha, \alpha_1, \dots, \alpha_{m-1}\}$, is sequentially continuous and the subtraction in $K_m(F)$ is sequentially continuous.

4.7. THEOREM.

(i) *$\Lambda_m(F)\cap VK_m(F)$ coincides with $\cap_{l\geqslant 1} lVK_m(F)$.*

(ii) *Let l be any prime number if k_0 is algebraic over $\mathbb{F}_p$ and $l = p$ otherwise. If F contains a primitive lth root ζ_l of unity and $lx = 0$ for $x \in K_m^{\mathrm{top}}(F)$, then $x = \{\zeta_l\}\cdot y$ for some $y \in K_{m-1}^{\mathrm{top}}(F)$.*

(iii) *The group $\cap_{l\geqslant 1} lVK_m(F)$ is divisible. The sequence*

$$0 \to \Lambda_m(F)\cap VK_m(F) \to VK_m(F) \to VK_m^{\mathrm{top}}(F) \to 0$$

splits.

(iv) *If k_0 is algebraic over $\mathbb{F}_p$ then (i) and (iii) hold if $\Lambda_m(F)\cap VK_m(F)$ is replaced by $\Lambda_m(F)$ and $\cap_{l\geqslant 1} lVK_m(F)$ by $\cap_{l\geqslant 1} lK_m(F)$.*

Proof. ***To deduce (i)*** we use Remark of 4.2 which implies that $\cap_{l\geqslant 1} lVK_m(F) \subset \Lambda_m(F)$. In view of Theorem 4.5 it remains to treat the case of fields of characteristic 0; there it is sufficient to consider the case of a field F of characteristic 0 with residue field k_{n-1} of characteristic p.

In the course of the proof we use Bloch–Kato's theorem which says that the norm residue homomorphism for a henselian discrete valuation field is an isomorphism [1].

To complete the proof of (i) we show by induction on r that $p^r VK_m(F)$ is equal to the intersection of open subgroups of $VK_m(F)$ which contain $p^r VK_m(F)$; then certainly $p^r VK_m(F) \supset \Lambda_m(F) \cap VK_m(F)$ (note that $lVK_m(F) = VK_m(F)$ for l prime to p).

First of all, we can assume that μ_p is contained in F applying the standard argument by using $(p, |F(\mu_p) : F|) = 1$ and l-divisibility of $VK_m(F)$ for l prime to p.

If $r = 1$, then we can use the description of $VK_m^{\text{top}}(F)/p$ provided by the Vostokov symbol of 3.2 and 4.3 and compare it with the description of the quotients

$$U_i K_m(F) + pK_m(F)/U_{i+1}K_m(F) + pK_m(F)$$

(where $U_i K_m(F)$ is generated by $1 + t_n^i \mathcal{O}_F$) provided by Bloch–Kato's theorem [1]; one deduces that $VK_m(F)/p = VK_m^{\text{top}}(F)/p$. From 4.3 we deduce then that the intersection of open subgroups in $VK_m(F)$ containing $pVK_m(F)$ is equal to $pVK_m(F)$.

It follows from the explicit description of the Vostokov symbol that

$$K_{n+1}(F)/p = K_{n+1}^{\text{top}}(F)/p \simeq \text{Hom}(G_{\text{ur,ab},p}, \mu_p).$$

If $\mu_{p^r} \subset F$, then one deduces by induction on r using Vostokov's primary elements $E(\chi)$ that

$$K_{n+1}(F)/p^r = K_{n+1}^{\text{top}}(F)/p^r \simeq \text{Hom}(G_{\text{ur,ab},p}, \mu_{p^r}).$$

Induction Step.

For a field F consider the pairing

$$(\ ,\ )_r \colon K_m(F)/p^r \times H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m}) \to H^{n+1}(F, \mu_{p^r}^{\otimes n})$$

given by the cup product and the map $F^* \to H^1(F, \mu_{p^r})$. If $\mu_{p^r} \subset F$, then from Bloch–Kato's theorem one can deduce that $(\ ,\ )_r$ is up to identifications the same as the Vostokov pairing $V_r(\ ,\ )$.

For $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$ put

$$A_\chi = \{\alpha \in K_m(F) : (\alpha, \chi)_r = 0\}.$$

Let α belong to the intersection of all open subgroups of $VK_m(F)$ which contain $p^r VK_m(F)$. Due to Lemma below $\alpha \in A_\chi$ for every $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$.

Set $L = F(\mu_{p^r})$ and $p^s = |L : F|$. From the induction hypothesis we deduce that $\alpha \in p^s VK_m(F)$ and hence $\alpha = N_{L/F}\beta$ for some $\beta \in VK_m(L)$. Then

$$0 = (\alpha, \chi)_{r,F} = (N_{L/F}\beta, \chi)_{r,F} = (\beta, i_{F/L}\chi)_{r,L}$$

where $i_{F/L}$ is the natural map. Keeping in mind the identification between the Vostokov pairing V_r and $(\ ,\ )_r$ for the field L we see that the element β is annihilated by $i_{F/L}K_{n+1-m}(F)$ with respect to the Vostokov pairing. From Lemma of 4.3 using its notation we deduce that

$$\beta \in (\sigma - 1)K_m(L) + p^{r-s} i_{F/L} K_m(F) + p^r K_m(L),$$

and therefore $\alpha \in p^r K_m(F)$. Since $K_m(F)/VK_m(F)$ has no nontrivial p-torsion (see 3.3), we conclude $\alpha \in p^r VK_m(F)$ as required.

Thus, the intersection of open subgroups containing $p^r VK_m(F)$ is equal to
$p^r VK_m(F)$ and $\Lambda_m(F) \cap VK_m(F) \subset p^r VK_m(F)$.

LEMMA. *A_χ is an open subgroup of $K_m(F)$.*

Proof. Due to Remark 4 of 4.6 it suffices to show that for every $b_2, \dots, b_m \in F^*$ the group

$$B = \{a \in F^* : (\{a, b_2, \dots, b_m\}, \chi)_r = 1\}$$

is open in F^*, i.e. if $a_i \to 1$ then $a_i \in B$ for all sufficiently large i. Denote by β the image of $\{b_2, \dots, b_m\}, \chi$ with respect to the pairing

$$K_{m-1}(F)/p^r \times H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m}) \to H^n(F, \mu_{p^r}^{\otimes n-1});$$

then B is the annihilator of β with respect to the pairing $F^*/p^r \times H^n(F, \mu_{p^r}^{\otimes n-1}) \to H^{n+1}(F, \mu_{p^r}^{\otimes n})$.

We argue by induction on r.

If $r = 1$, then using the link between $(\ ,\ )_1$ and $V_1(\ ,\ )$ and the explicit formula for V_1 we deduce that B is open.

Induction step. Let $r > 1$. Since $a_i^p \to 1$, the induction hypothesis implies that $a_i^p \in B$ for all sufficiently large i. So $(a_i, \beta) \in \mathrm{Tors}_p H^{n+1}(F, \mu_{p^r}^{\otimes n})$ for all sufficiently large i.

The exact sequence $1 \to \mu_p^{\otimes m} \to \mu_{p^r}^{\otimes m} \to \mu_{p^{r-1}}^{\otimes m} \to 1$ induces the exact sequence

$$H^n(F, \mu_{p^r}^{\otimes n}) \to H^n(F, \mu_{p^{r-1}}^{\otimes n}) \to H^{n+1}(F, \mu_p^{\otimes n}) \to H^{n+1}(F, \mu_{p^r}^{\otimes n}).$$

Due to Bloch–Kato's theorem the first map is surjective, hence the third one is injective. Therefore, $\mathrm{Tors}_p H^{n+1}(F, \mu_{p^r}^{\otimes n})$ is isomorphic to a quotient of $H^{n+1}(F, \mu_p^{\otimes n})$ which due to Bloch–Kato's theorem is isomorphic to a quotient S_1 of $K_{n+1}^{\mathrm{top}}(F)/p$. Similar to the previous argument $H^n(F, \mu_{p^r}^{\otimes n-1})/p$ is isomorphic to a subquotient S_2 of $K_n(F)/p$. It remains to show that the annihilator of the image of β in S_2 with respect to the pairing $F^*/p \times S_2 \to S_1$ is open; identifying the latter with the induced one by the Vostokov pairing one completes the induction step. □

To prove (ii) assume that the field F contains a primitive lth root of unity ζ_l. If $l \neq p$, then from Remark of 4.2 and property (3) in 4.1 we deduce the result.

Suppose that $l = p$.

The exact sequence $1 \to \mu_{p^s}^{\otimes m} \to \mu_{p^{s+1}}^{\otimes m} \to \mu_p^{\otimes m} \to 1$ induces the commutative diagram

$$\begin{array}{ccccc}
\mu_p \otimes K_{m-1}(F)/p & \longrightarrow & K_m(F)/p^s & \xrightarrow{p} & K_m(F)/p^{s+1} \\
\downarrow & & \downarrow & & \downarrow \\
H^{m-1}(F, \mu_p^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^s}^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^{s+1}}^{\otimes m})
\end{array}$$

and the bottom horizontal sequence is exact, the left and the right vertical homomorphisms are isomorphisms due to Bloch–Kato's theorem. Hence if $px \in p^r K_m(F)$, then $x = \{\zeta_p\} \cdot a_{r-1} + p^{r-1} c_{r-1}$. Denote by D_r the preimage of the closed subgroup $p^r K_m^{\mathrm{top}}(F)$ with respect to the continuous homomorphism $K_{m-1}^{\mathrm{top}}(F) \xrightarrow{h} VK_m^{\mathrm{top}}(F)$, $z \to \{\zeta_p\} \cdot z$. The kernel of h is equal to $D = \cap D_r$.

Let $\alpha \in K_{m-1}^{\mathrm{top}}(F) \setminus D$. There is a positive integer r such that $h(\alpha) \notin p^r VK_m^{\mathrm{top}}(F)$. Include α in a topological basis of $K_{m-1}^{\mathrm{top}}(F)$ modulo p (see 4.3 and Remark 3 of 4.1). Let $U_\alpha + pK_{m-1}^{\mathrm{top}}(F)$ be the subgroup of $K_{m-1}^{\mathrm{top}}(F)$ topologically generated by elements of the basis distinct from multiples of α. Then $a_{r-1} \in D_r \subset U_\alpha$. Since $\{U_\alpha\}$ is a basis of open neighbourhoods of 0 with respect to the quotient topology of ν_{m-1} extended naturally to $K_{m-1}^{\mathrm{top}}(F)/D$ (see Remark 2 of 4.5 and Remark 3 of 4.1) we deduce that $\{a_r \bmod D\}$ is a Cauchy sequence in $K_{m-1}^{\mathrm{top}}(F)/D$. Since $D \supset pK_{m-1}^{\mathrm{top}}(F)$ there is a closed subgroup D' of $K_{m-1}^{\mathrm{top}}(F)$ such that $D/p \oplus D'/p = K_{m-1}^{\mathrm{top}}(F)/p$. Let y be a limit of the image of $\{a_r\}$ in D'/p (which exists since D' is complete) then $z = \{\zeta_p\} \cdot y$.

To prove (iii) note that for l' prime to p Theorem 4.6 (i) shows that $VK_m^{\mathrm{top}}(F)$ has no nontrivial l'-torsion; hence by part (i) $\cap_{l\geqslant 1} lVK_m(F)$ is l'-divisible.

Let $z \in \cap_{l\geqslant 1} lVK_m(F)$. Then $z = px$ for some $x \in VK_m(F)$. Suppose that a primitive pth root of unity belongs to F. Then from the previous part we get $x = \{\zeta_p\} \cdot y + w$ with $w \in \Lambda_m(F) \cap VK_m(F)$. Hence $z = pw$ and $\Lambda_m(F) \cap VK_m(F)$ is p-divisible. If F doesn't contain a primitive pth root, then pass to $F(\zeta_p)$ and use the norm map argument.

(iv) follows from Remark 1 of 4.5. □

REMARK TO THE PROOF. Let O_F be the discrete valuation ring with respect to t_n. Using Kurihara's exponential map [25]

$$\Omega_{O_F}^{m-1} \to K_m(F)/p^r K_m(F),$$

whose definition uses the syntomic complex, and the description by Bloch and Kato of $U_i K_m(F) + p^r K_m(F)/U_{i+1}K_m(F) + p^r K_m(F)$ for small i one can prove that $p^r VK_m(F)$ is equal to the intersection of open subgroups of $VK_m(F)$ which contain $p^r VK_m(F)$ in the same way as in the case $r = 1$.

COROLLARY. *If k_0 is algebraic over $\mathbb{F}_p$ then for every integer $l \geqslant 1$*

$$K_m(F)/l \simeq K_m^{\mathrm{top}}(F)/l.$$

In general

$$VK_m(F)/l \simeq VK_m^{\mathrm{top}}(F)/l.$$

Thus, one can use topological generators of $K_m^{\mathrm{top}}(F)/l$ to describe the structure of $K_m(F)/l$. In particular, if F contains a primitive lth root of unity or l is prime or l is a power of char(F)*, then all relations between topological generators of $VK_m^{\mathrm{top}}(F)/l$ are known (as an application of the Artin–Schreir–Witt–Parshin or Vostokov pairing).*

REMARK 1. Let F be of characteristic zero. The l-adic symbol defined in [18]

$$K_n(F) \to \prod_l H^n(F, \mathbb{Z}_l(n))$$

induces the monomorphism

$$VK_n^{\mathrm{top}}(F) \to \prod_l H^n(F, \mathbb{Z}_l(n)).$$

REMARK 2. Let ρ_m be the finest topology on $K_m(F)$ for which the map from $F^{*\oplus m}$ to $K_m(F)$ is sequentially continuous and the intersection of all neighbourhoods of zero in $K_m(F)$ contains $\cap_{l\geqslant 1} lK_m(F)$ (the topology ρ_m was used in [2–3]). Then ρ_m is $\geqslant \lambda_m$ and the intersection of all neighbourhoods of zero in $K_m(F)$ with respect to ρ_m coincides with $\cap_{l\geqslant 1} lK_m(F)$. On the level of subgroups λ_m and ρ_m coincide.

REMARK 3. For another approach to describe the set of *open subgroups* of $K_m(F)$ see [9].

REMARK 4. For char$(F) = 0$ I. B. Zhukov found (applying higher class field theory) a complete algebraic description of $K_n^{\mathrm{top}}(F)$ in several cases (see [21]). In particular, if $T_p K_n^{\mathrm{top}}(F)$ is the topological closure of the p-torsion in $K_n^{\mathrm{top}}(F)$ and F has a local parameter t_n algebraic over $\mathbb{Q}_p$, then $VK_n^{\mathrm{top}}(F)/T_p K_n^{\mathrm{top}}(F)$ possesses a topological basis of the form $\{\varepsilon, t_{n-1}, \dots, t_1\}$ with ε

running free $\mathbb{Z}_p$-generators of the group of principal units of the algebraic closure of $\mathbb{Q}_p$ in F modulo its p-primary torsion.

REMARK 5. Let k_0 be finite (resp. infinite and not p-algebraically closed). From the point of view of higher local class field theory Theorems 4.6 and 4.7 show that the kernel of the reciprocity map

$$K_n(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

(resp.

$$VK_n(F) \to \mathrm{Hom}(G_{\mathrm{ur,ab},p}, \mathrm{Gal}(E/F))$$

where E is a maximal abelian totally ramified p-extension of F), which due to existence theorem [9], [2–3] coincides with the intersection of all open subgroups of finite index in $K_n(F)$ (resp. normic subgroups of $VK_n(F)$ [4, sect.5]), is equal to $\Lambda_n(F) = \cap_{l\geqslant 1} lK_n(F)$ (resp. $\Lambda_n(F) \cap VK_n(F) = \cap_{l\geqslant 1} lVK_n(F)$). Thus, the induced map from $K_n^{\mathrm{top}}(F)$ (resp. from $VK_n^{\mathrm{top}}(F)$) is a monomorphism (with dense image).

REMARK 6. It is an open problem to describe the torsion in $\Lambda_m(F)$. For $n = 1$ see [11] (finite k_0), [12] (perfect k_0).

4.8. The norm map on topological K-groups. It follows easily from 4.1 that for a cyclic extension L/F of a prime degree $K_m^{\mathrm{top}}(L)$ is generated by L^* and the image of $K_{m-1}^{\mathrm{top}}(F)$.

For an arbitrary multidimensional local field define the norm on $K^{\mathrm{top}}(F)$ as induced from the norm on Milnor K-groups.

Using Theorems 4.6–4.7, Remark 4 of 4.6 and a description of $N_{L/F}\colon L^* \to F^*$ for a cyclic extension of prime degree [2–3] one directly shows that for a finite extension M/F

(1) the image of an open subgroup in $K_n(M)$ with respect to the norm map is an open subgroup,
(2) the preimage of an open subgroup in $K_m(F)$ with respect to the norm map is an open subgroup.

REMARK. In characteristic p if k_0 is finite there is a very simple way to define the norm map on topological K-groups [15–16]:

(a) for a cyclic extension L/F of a prime degree introduce $N_{L/F}\colon K_m^{\mathrm{top}}(L) \to K_m^{\mathrm{top}}(F)$ as induced by the norm on K_1;
(b) for an arbitrary abelian extension L/F define the norm presenting L/F as a tower of cyclic extensions of prime degree.

Correctness of this definition follows from an application of the Artin–Schreier–Witt–Parshin and tame pairings [15]. Compatibility of the just defined norm with induced from the Milnor K-groups follows then from 4.7.

THEOREM. *If L/F is a cyclic extensions of a prime degree l with a generator σ then the sequence*

$$K_n^{\mathrm{top}}(F)/l \oplus K_n^{\mathrm{top}}(L)/l \xrightarrow{i_{F/L}\oplus(1-\sigma)} K_n^{\mathrm{top}}(L)/l \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)/l$$

is exact, where $i_{F/L}$ is induced by the field embedding.

This theorem is verified by explicit calculations in K_n^{top}/l-groups whose structure is completely known due to an application of the Artin–Schreier–Witt–Parshin, tame and Vostokov pairings (adjoin if necessary a primitive lth root of unity ζ). Similar calculations show that the index of the norm group $N_{L/F}K_n^{\mathrm{top}}(L)$ in $K_n^{\mathrm{top}}(F)$ is finite of order $|L:F|$ if the latter is prime and k_0 is finite [2–3].

It is well known that a corollary of this theorem and the description of the torsion of $K_n^{\text{top}}(F)$ of 4.7 is the exactness of the sequence

$$K_n^{\text{top}}(L) \xrightarrow{1-\sigma} K_n^{\text{top}}(L) \xrightarrow{N_{L/F}} K_n^{\text{top}}(F)$$

for every cyclic extension L/F of prime degree if k_0 is finite and of degree p in the case of perfect k_0.

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Appendix: On the group $K_2(F)/\bigcap_{l\geqslant 1} lK_2(F)$

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Abstract. We construct a field F containing a primitive pth root and such that p-torsion of the group $K_m(F)/\cap_{l\geqslant 1} lK_m(F)$ is not generated by p-torsion in F^*. The method of proof is to work with the field of rational functions of an infinite product of certain Severi–Brauer varieties using Merkur'ev–Suslin's theorem, Suslin's theorem on the torsion in $K_2(F)$ and Kahn's theorem.

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Key words. Merkur'ev–Suslin theorem, Severi–Brauer variety, generic splitting field.

1. Introduction

Throughout the text we denote by μ_m the group of mth roots of unity. For an abelian group A we denote m-torsion points of A by $\operatorname{Tors}_m A$.

Fix a prime integer p. In this appendix F is a field of characteristic different from p. In the case $p = 2$ we will suppose that $\sqrt{-1} \in F$. Let ζ_{p^i} be a primitive p^ith root and $F_i = F(\zeta_{p^i})$, $F_\infty = \cup F_i$. For a field F put $s(F) = \sup\{n : F_n = F\}$.

1.1. Definition. Set

$$DK_n(F) = \cap_{l\geqslant 1} lK_n(F), \quad D_pK_n(F) = \cap_i p^i K_n(F), \quad K_n^t(F) = K_n(F)/DK_n(F).$$

Clearly, $K_*^t(F) = \oplus_{n=0}^{\infty} K_n^t(F)$ is a graded ring. There is a natural question, for which fields the structure of the torsion subgroup in $K_n^t(F)$ is "standard". More precisely, *for which fields the condition $\zeta_{p^n} \in F^*$ implies that $\operatorname{Tors}_{p^n} K_n^t(F) = \{\zeta_{p^n}\} \cdot K_{n-1}^t(F)$?*

Section 4 of the previous paper contains a description of $K_m^t(F) = K_m^{\text{top}}(F)$ for a higher local field F; in particular it is shown that the answer to the question in the previous paragraph is positive for higher local fields with finite residue field k_0.

The main purpose of this note is to construct fields for which the answer is negative. We prove the following

1.2. Theorem. *Let p be a prime integer. Then there exists a field F such that $\zeta_p \in F$ and* $\operatorname{Tors}_p K_2^t(F) \neq \{\zeta_p, F^*\}$.

The proof is given in 3.4; now we deduce some corollaries.

Corollary 1. *Let p be a prime integer and $m \geqslant 2$. Then there exists a field F such that $\zeta_p \in F$ and* $\operatorname{Tors}_p K_m^t(F) \neq \{\zeta_p\}K_{m-1}^t(F)$.

Proof. Let F_0 be a field satisfying the conditions of the theorem. Let $u_0 \in \mathrm{Tors}_p\, K_2^t(F_0)$ be such that $u_0 \notin \{\zeta_p, F^*\}$. Setting $F = F_0(t_1, \ldots, t_{m-2})$ and $u = u_0 \cdot \{t_1, \ldots, t_{m-2}\}$ one can see that $u \in \mathrm{Tors}_p\, K_m^t(F)$ and $u \notin \{\zeta_p\} \cdot K_{m-1}^t(F)$. Hence, $\mathrm{Tors}_p\, K_m^t(F) \neq \{\zeta_p\}K_{m-1}^t(F)$. □

COROLLARY 2. *For all $m \geqslant 2$ there exists a cyclic field extension L/F of degree p with Galois group* $\mathrm{Gal}(L/F) = \{1, \sigma, \ldots, \sigma^{p-1}\}$ *such that the sequence*

$$K_m^t(L) \xrightarrow{1-\sigma} K_m^t(L) \xrightarrow{N_{L/F}} K_m^t(F)$$

is not exact.

Proof. Suppose that the sequence is exact for all cyclic extensions L/F. Then arguments of [3, Lemma 10.4] show that $\mathrm{Tors}_p\, K_m^t(F) = \{\zeta_p\}K_{m-1}^t(F)$ for all fields F containing ζ_p. We get a contradiction to Corollary 1. □

First we consider the notion of splitting fields.

2. Generic splitting fields for elements of the group $K_2(F)/p^n$

2.1. Denote by h_i the homomorphism

$$h_i\colon K_2(F) \to \mathrm{Tors}_{p^i}\, \mathrm{Br}(F_i), \qquad \{a, b\} \mapsto (a, b)_{\zeta_{p^i}}.$$

In what follows we will use the following

PROPOSITION. *The kernel of h_i is equal to $p^i K_2(F)$.*

Proof. Using Merkur'ev–Suslin's theorem [3] the injectivity of

$$K_2(F)/p^i \to \mathrm{Tors}_{p^i}\, \mathrm{Br}(F_i)$$

is equivalent to the injectivity of $H^2(F, \mu_{p^i}^{\otimes 2}) \to H^2(F(\mu_{p^i}), \mu_{p^i}^{\otimes 2})$. The latter follows from Kahn's theorem [2, Th.1(1)]. □

LEMMA. *Let $u \in K_2(F)$ and let $j \geqslant i \geqslant 1$. Then $(h_i(u))_{F_j} = p^{j-i}h_j(u)$. Moreover, if $j \geqslant i \geqslant s(F)$, then $N_{F_j/F_i}(h_j(u)) = h_i(u)$.*

2.2. DEFINITION. Let $u \in K_2(F)$ and n be a positive integer. A variety X is said to be a (u, n)-generic if the following conditions hold:

(1) X is a homogeneous variety,

(2) for a field extension E/F the following conditions are equivalent:

  (a) $u_E \in p^n K_2(E)$,

  (b) variety X_E is rational.

REMARK. Since X is a homogeneous variety, the property of X_E to be rational is equivalent to the existence of a rational point on the E-variety X_E.

LEMMA. *Let $u \in K_2(F)$ and n be a positive integer. Then*

(1) *Suppose that X is a (u,n)-generic. Then $u_{F(X)} \in p^n K_2(F(X))$.*

(2) *Suppose that X is a (u,n)-generic. Let L/F be a field extension. Then X_L is a (u_L,n)-generic.*

(3) *Suppose that the X_1 and X_2 are (u,n)-generic. Then the extension $F(X_1)/F$ is stably equivalent to extension $F(X_2)/F$.*

Proof. (1) $X_{F(X)}$ has a rational point and hence condition (b) in the definition holds for the field $E = F(X)$. Hence, condition (a) holds.

(2) Obvious.

(3) By the previous definition and (1), varieties $(X_2)_{F(X_1)}$ and $(X_1)_{F(X_2)}$ are rational. Therefore the extensions $F(X_1 \times X_2)/F(X_1)$ and $F(X_1 \times X_2)/F(X_1)$ are purely transcendental. Hence, the extension $F(X_1)/F$ is stably equivalent to extension $F(X_2)/F$. □

2.3. The following example proves the existence of (u,n)-genetic varieties (recall that in the case $p = 2$ we suppose that $i \in F^*$; if $p = 2$ and $i \notin F^*$ we do not know whether there exists a (u,n)-generic variety).

EXAMPLE. *Let $u \in K_2(F)$ and let A be a central simple F_n-algebra such that $[A] = h(u_{F_n}) \in$ $\mathrm{Tors}_{p^n}\,\mathrm{Br}(F_n)$. Let S be the Severi–Brauer variety of A* [3]. *Then the variety $R_{F_n/F}(S)$ is a (u,n)-generic.*

Proof. Let E/F be an extension. Since F_n/F is a Galois extension, there exists an isomorphism of F-algebras $E \otimes_F F_n \simeq \prod E_n$. Properties of Weil restriction show that

$$\begin{aligned}\mathrm{mor}_F(\mathrm{Spec}(E), R_{F_n/F}(S)) &= \mathrm{mor}_{F_n}(\mathrm{Spec}(E \otimes_F F_n), S)\\ &= \mathrm{mor}_{F_n}\Bigl(\coprod \mathrm{Spec}(E_n), S\Bigr) = \coprod \mathrm{mor}_{F_n}(\mathrm{Spec}(E_n), S).\end{aligned}$$

Therefore the variety $(R_{F_n/F}(S))_E$ has a rational point if and only if S_{E_n} has a rational point. Since S_{E_n} is the Severi–Brauer variety of A_{E_n}, the variety S_{E_n} has a rational point if and only if the algebra A_{E_n} splits. By Proposition 2.1 this is equivalent to $u_E \in p^n K_2(E)$. □

2.4. PROPOSITION. *Let $u \in K_2(F)$ and let X be a (u,n)-generic variety. Then $\mathrm{Br}(F(X)/F)$ is generated by $h_r(u)$ where $r = \min(n, s(F))$.*

Proof. Let A be an F_n-algebra corresponding to the element $h_n(u) \in \mathrm{Tors}_{p^n}\,\mathrm{Br}(F_n)$. Taking into account Lemma of 2.2 and Example of 2.3, we can assume that $X = R_{F_n/F}(S)$ where S is the Severi–Brauer variety of A. It is well known that the group $\mathrm{Br}(F(X)/F)$ is generated by the element $N_{F_n/F}([A])$ [3]. Hence $\mathrm{Br}(F(X)/F)$ is generated by $N_{F_n/F}(h_n(u))$. If $n \leqslant s(F)$, we have $F_n = F$, $r = \min(n, s(F)) = n$ and

$$N_{F_n/F}(h_n(u)) = h_n(u) = h_r(u).$$

If $n > s = s(F)$, then $r = \min(n, s) = s$ and $N_{F_n/F}(h_n(u)) = N_{F_n/F_r}(h_n(u)) = h_r(u)$. □

COROLLARY 1. *Let $u \in K_2(F)$ and let X be a (u,n)-generic variety. Then for any $m \geqslant n$ the kernel of the homomorphism $K_2(F)/p^m \to K_2(F(X))/p^m$ is generated by the element $p^{m-n}u$.*

Proof. Obviously, the element $p^{m-n}u$ lies in the kernel. By the previous proposition the group $\mathrm{Br}(F_m(X)/F_m)$ is generated by $h_r(u)$, where $r = \min(n, s(F_m))$. Since $m \geqslant n$, we have $r = n$. So $h_r(u) = h_n(u) = h_m(p^{n-m}u)$. Since the homomorphism h_m is injective, the proof is complete. □

COROLLARY 2. *Let $u \in K_2(F)$ and let X be a (u, n)-generic variety. Then for every m satisfying the condition $s(F) \leqslant m \leqslant n$, the kernel of the homomorphism $K_2(F)/p^m \to K_2(F(X))/p^m$ is generated by the element u.*

Proof. Obviously, the element u lies in the kernel. By the proposition the group $\mathrm{Br}(F_m(X)/F_m)$ is generated by $h_r(u)$, where $r = \min(n, s(F_m))$. Since $s(F) \leqslant m \leqslant n$, we have $r = m$. Hence $h_r(u) = h_m(u)$. Since the homomorphism h_m is injective, the proof is complete. □

3. On the factorgroup $K_2(F)/\cap_{l\geqslant 1} lK_2(F)$

Now we return to the Theorem of 1.2. The following statement is easy.

3.1. LEMMA. *Let A be an abelian group such that the homomorphism*

$$A/p^n A \to p^m A/p^{n+m} A, \quad \overline{a} \mapsto \overline{p^m a}$$

is bijective for all n, m. Then the group $D_p(A) = \cap_n p^n A$ is p-divisible and the factor group $A/D_p(A)$ is torsion free.

3.2. LEMMA. *Suppose that $s(F) = \infty$. Then*

(1) *the group $D_p(F)$ is p-divisible*

(2) *the factor group $K_2(F)/D_pK_2(F)$ is torsion free.*

(3) *for any finitely generated extension L/F the homomorphism*

$$\alpha\colon K_2(F)/D_pK_2(F) \to K_2(L)/D_pK_2(L)$$

is injective.

Proof. (1), (2) follow from Lemma 3.1 and the following claim: Let E be a field such that $\zeta_{p^n} \in E$. Then for any $m \leqslant n$ the homomorphism

$$K_2(E)/p^m K_2(E) \to p^{n-m} K_2(E)/p^n K_2(E), \quad \overline{u} \mapsto \overline{p^{n-m} u}$$

is an isomorphism.

To show the claim, let $u \in K_2(E)$ be such that $p^{n-m}u$ is divisible by p^n. We need to prove that u is divisible by p^m. By assumption, there exists $w \in K_2(E)$ so that $p^{n-m}u = p^n w$. Hence, $p^{n-m}(u - p^m w) = 0$. Therefore, $u \in \mathrm{Tors}_{p^{n-m}} K_2(E) + p^m w$. Since $\mathrm{Tors}_{p^{n-m}} K_2(E) = \{\zeta_{p^{n-m}}, E^*\} = p^m\{\zeta_{p^n}, E^*\} \subset p^m K_2(E)$ (the first equality is Suslin's result of [4]) it follows that $u \in p^m K_2(E)$.

(3) It suffices to consider only two cases: $L = F(t)$ is the field of rational functions; L/F is finite extension. The case $L = F(t)$ is obvious (one can use specialization arguments). If L/F is finite extension, the composite

$$K_2(F) \to K_2(L) \xrightarrow{N_{L/F}} K_2(F)$$

coincides with multiplication by $|L : F|$. Hence the kernel of α lies in the torsion subgroup of $K_2(F)/D_p(F)$. Since the group $K_2(F)/D_p(F)$ is torsion free, the proof is completed. □

COROLLARY. *Let $u \in K_2(F)$ be such that $u_{F_\infty} \notin D_pK_2(F_\infty)$ and let L/F be an arbitrary finitely generated extension. Then $u_{LF_\infty} \notin D_pK_2(LF_\infty)$.*

3.3. Before we state our next assertion, we introduce the following notation.

Let X_i, $(i = 1, 2, \dots)$ be an infinite collection of smooth F-varieties. Let us denote by $X_{\leqslant n}$ the variety $X_1 \times \cdots \times X_n$. By $X_{\leqslant\infty}$ we denote the infinite product

$$\prod_i X_i = X_1 \times \cdots \times X_i \times \dots$$

In other words, $X_{\leqslant\infty}$ is the inverse limit of varieties $X_{\leqslant n}$ (note that $X_{\leqslant\infty}$ is not a variety except for the case $\sum \dim X_i < \infty$).

Thus, $F(X_{\leqslant\infty})$ coincides with the direct limit of the fields $F(X_{\leqslant n})$. By $X_{>n}$ we denote the product

$$\prod_{i>n} X_i = X_{n+1} \times X_{n+2} \times \dots$$

Obviously, we have $X_{\leqslant\infty} = X_{\leqslant n} \times X_{>n}$.

PROPOSITION. *Let F be a field such that $s = s(F) \neq \infty$. Let $u \in K_2(F)$ be such that $u_{F_\infty} \notin D_pK_2(F_\infty)$. Then there exists a field extension E/F such that $pu_E \in D_pK_2(E)$ and u_E does not belong to $D_pK_2(E) + \operatorname{Tors} K_2(E)$.*

Proof. Let X_i be a (pu, i)-generic variety $(i \geqslant 1)$. We let $E = F(X_{\leqslant\infty})$. The definition of X_i and Lemma 2.2 show that $pu_{F(X_i)} \in p^iK_2(F(X_i))$. Hence $pu_E \in p^iK_2(E)$ for all $i \geqslant 1$. Therefore, $pu_E \in D_pK_2(E)$. Now we need to verify that $u_E \notin D_pK_2(E) + \operatorname{Tors} K_2(E)$.

Suppose that $u_E \in D_pK_2(E) + \operatorname{Tors} K_2(E)$. Let $u_E = \mu + \gamma$ be such that $\mu \in D_pK_2(E)$ and $\gamma \in \operatorname{Tors} K_2(E)$. Let r be the order of the element γ. By [1], r is prime to char(F). Adding to all fields the element ζ_r, we can assume that $\zeta_r \in F^*$. Then the element γ has the form $\{\zeta_r, z\}$ where $z \in E^*$. Then, $\gamma_{E(\sqrt[r]{z})} = \{\zeta_r, z\}_{E(\sqrt[r]{z})} = 0$. Therefore, $u_{E(\sqrt[r]{z})} = \mu_{E(\sqrt[r]{z})} \in D_pK_2(E(\sqrt[r]{z}))$.

Let n be such that $z \in F(X_{\leqslant n})$ and let $K = F(X_{\leqslant n})(\sqrt[r]{z})$. We have $E(\sqrt[r]{z}) = F(X_{\leqslant n} \times X_{>n})(\sqrt[r]{z}) = K(X_{>n})$. Hence, $u_{K(X_{>n})} \in D_pK_2(K(X_{>n}))$.

By Corollary 3.2 we have $u_{KF_\infty} \notin D_pK_2(KF_\infty)$. Let m be any integer satisfying two conditions $m \geqslant s(F)$ and $u_{KF_\infty} \notin p^mK_2(KF_\infty)$. Adding the element ζ_{p^m} to all fields, we can assume that $m = s(F)$ and $u_K \notin p^mK_2(K)$.

We have $u_{K(X_{>n})} \in D_pK_2(K(X_{>n})) \subset p^mK_2(K(X_{>n}))$. From $u_K \notin p^mK_2(K)$ we deduce that there exists k such that $u_{K(X_{n+1}\times\cdots\times X_k)}$ is divisible by p^m. However $u_{K(X_{n+1}\times\cdots\times X_{k-1})}$ is not divisible by p^m. Setting $\widetilde{K} = K(X_{n+1} \times \cdots \times X_{k-1})$, we have $u_{\widetilde{K}} \notin p^mK_2(\widetilde{K})$, $u_{\widetilde{K}(X_k)} \in p^mK_2(\widetilde{K}(X_k))$. Since $(X_k)_{\widetilde{K}}$ is a $(pu_{\widetilde{K}}, k)$-generic, Corollaries 1 and 2 of 2.4 show that $u_{\widetilde{K}}$ is divisible by $pu_{\widetilde{K}}$ in the group $K_2(\widetilde{K})/p^m$. Hence there exists an integer r such that $(u - rpu)_{\widetilde{K}} \in p^mK_2(\widetilde{K})$. Since $1 - rp$ is invertible modulo p^m, we have $u_{\widetilde{K}} \in p^mK_2(\widetilde{K})$. We get a contradiction. □

3.4. *Proof of Theorem 1.2.*

Let F be a field and an element $a \in F^*$ be such that

$1 \leqslant s(F) < \infty$,

$a \notin F_\infty^{*p}$,

$a \in F^{*m}$ for all integer m prime to p.

It is not difficult to construct fields with the required properties. For example one can take

$$F = \mathbb{Q}(\zeta_p)(x)(\{\sqrt[m]{x} : m \text{ runs over all integers prime to } p\}), \qquad a = x.$$

Another example is $F = \mathbb{Q}_p(\zeta_p)$, $a = 1 + p(1 - \zeta_p)$ (condition $a \notin F_\infty^{*p}$ holds because $F(\sqrt[p]{a})/F$ is unramified and F_n/F is totally ramified for all n).

Now, let $\widetilde{F} = F(t)$ and $u = \{a, t\} \in K_2(\widetilde{F})$. Clearly $u_{\widetilde{F}_\infty} \notin pK_2(\widetilde{F}_\infty)$. Indeed, otherwise $a = \partial_t(\{a, t\}) \in F_\infty^{*p}$, and we get a contradiction. Thus, allthe conditions of Proposition 3.3 hold for the field $\widetilde{F}$ and element u. Let $E/\widetilde{F}$ be a field extension as in Proposition 3.3. Then $pu_E \in D_pK_2(E)$. Hence $pu_E \in mK_2(E)$ for all m which are a power of p. If m is prime to p, we have $u_E = \{a, t\} \in \{F^{*m}, t\} \subset mK_2(E)$. Therefore $pu_E \in mK_2(E)$ for all m. Hence $pu_E \in DK_2(E)$. Therefore, $u_E \in \mathrm{Tors}_p\, K_2^t(E)$. Assume that in the group $K_2^t(E)$, we had $u = \{\zeta_p, z\}$. Then in the group $K_2(E)$ we would have $u \in \{\zeta_p, z\} + DK_2(E) \subset \mathrm{Tors}\, K_2(E) + D_pK_2(E)$. This contradicts the conditions on E stated in Proposition 3.3. □

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