CHAPTER I

ADELES OVER $\mathbb{Q}$

§1.1 Absolute values

Definition 1.1.1 (Absolute value) *An absolute value on a field F is a non-negative real valued function $|\,|$ on F which satisfies the conditions:*

(i) $|x| = 0$ *if and only if* $x = 0$,
(ii) $|xy| = |x| \cdot |y|$,
(iii) $|x + y| \le |x| + |y|$, *(triangle inequality)*

for all $x, y \in F$.

If an absolute value $|\,|$ on a field F satisfies the stronger condition

$$|x + y| \le \max\big(|x|, |y|\big), \tag{1.1.2}$$

then it is called a non-archimedean absolute value. If condition (1.1.2) fails for some $x, y \in F$, then $|\,|$ is called an archimedean absolute value.

It is always possible to define a trivial absolute value $|\,|_{\text{trivial}}$ on any field F where

$$|x|_{\text{trivial}} = \begin{cases} 1, & \text{if } x \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $|\,|_{\text{trivial}}$ is not very interesting, we shall usually exclude it in our discussions.

Definition 1.1.3 (Equivalence of Absolute values) *Two absolute values $|\,|_1$ and $|\,|_2$, defined on the same field F, are termed equivalent if there exists $c > 0$ such that $|x|_1 = |x|_2^c$ for all $x \in F$.*

Example 1.1.4 The field $\mathbb{Q}$ of rational numbers has the classical (and very ancient) archimedean absolute value which we denote by $|\,|_\infty$ which is defined by

$$|x|_\infty = \begin{cases} x, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0, \end{cases} \tag{1.1.5}$$

for all $x \in \mathbb{Q}$. For each prime p one may define the non-archimedean absolute value $|\,|_p$ as follows. Given $x \in \mathbb{Q}$ with $x = p^k \cdot \frac{m}{n}$ with $p \nmid mn$, and $k \in \mathbb{Z}$, we define

$$|x|_p = \left| p^k \cdot \frac{m}{n} \right| = p^{-k}. \tag{1.1.6}$$

The definition of $|\,|_p$ has the effect that the non-archimedean absolute values of numbers divisible by high powers of p become small.

Theorem 1.1.7 (Ostrowski) *The only non-trivial absolute values on $\mathbb{Q}$ are those equivalent to the $|\,|_p$ or the ordinary absolute value $|\,|_\infty$.*

Proof: See [Cassels, 1986], [Murty, 2002].
□

Theorem 1.1.8 (Product formula) *Let $\alpha \in \mathbb{Q}$ with $\alpha \neq 0$. The absolute values $|\,|_v$, given by (1.1.5), (1.1.6), satisfy the product formula*

$$\prod_v |\alpha|_v = 1$$

where the product is taken over all $v \in \{\infty, 2, 3, 5, 7, 11, 13, \dots\}$, i.e., $v = \infty$ or v is a prime.

Proof: The proof is elementary and left to the reader.
□

Definition 1.1.9 (Finite and infinite primes) *Following the modern tradition we shall call $v = 2, 3, 5, 7, 11, 13 \dots$ the finite primes and $v = \infty$ the "infinite or archimedean prime." Henceforth, we shall adhere to the convention that v refers to an arbitrary prime v (with v finite or infinite) while p refers specifically to a finite prime.*

§1.2 The field $\mathbb{Q}_p$ of p-adic numbers

An absolute value $|\,|$ on a field F allows us to define the notion of distance between two elements $x, y \in F$ as $|x - y|$. We may also introduce a topology on F where the basis of open sets consists of the open balls $B_r(a)$ with center $a \in F$ and radius $r > 0$:

$$B_r(a) = \big\{x \;\big|\; |x - a| < r\big\}.$$

A sequence of elements $x_1, x_2, x_3, \dots \in F$ is termed Cauchy provided

$$|x_m - x_n| \longrightarrow 0 \qquad (m, n \to \infty). \tag{1.2.1}$$

A field F with a non-trivial absolute value $|\,|$ is said to be complete if all Cauchy sequences of elements $x_1, x_2, x_3, \dots \in F$ have the property that there exists an element $x^* \in F$ such that $|x_n - x^*| \to 0$ as $n \to \infty$, i.e., all Cauchy sequences converge.

If a field F is not complete, it is possible to complete it by standard methods of analysis. In brief, one adjoins to the incomplete field F all the elements arising from equivalence classes of Cauchy sequences, where two Cauchy sequences $\{x_1, x_2, \dots\}$, $\{y_1, y_2, \dots\}$ are equivalent if $\lim_{i\to\infty} |x_i - y_i| = 0$. The original elements $\alpha \in F$ are then realized as the equivalence class of the constant Cauchy sequence $\{\alpha, \alpha, \alpha, \dots\}$. Addition, subtraction, and multiplication of the representatives $\{x_i\} = \{x_1, x_2, \dots\}$, $\{y_i\} = \{y_1, y_2, \dots\}$ of two equivalence classes of Cauchy sequences are defined by

$$\{x_i\} \pm \{y_i\} = \{x_i \pm y_i\}, \qquad \{x_i\} \cdot \{y_i\} = \{x_i \cdot y_i\}.$$

The definition of division is the same, except one has to be careful to not divide by zero because in a Cauchy sequence $\{x_1, x_2, x_3, \dots\}$, some of the x_i may be 0. Happily, this is not a problem, because every Cauchy sequence is equivalent to a Cauchy sequence without any zero terms and we always choose such a representative for performing division. The sequence of quotients will be Cauchy, provided the Cauchy sequence by which we divide does not converge to zero.

Definition 1.2.2 (p-adic fields) *Let p be a prime number. The completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\,|_p$, defined by (1.1.6), is denoted as $\mathbb{Q}_p$ and called the p-adic field.*

We now present two explicit constructions of $\mathbb{Q}_p$.

Analytic construction of $\mathbb{Q}_p$: The first construction we present is based on the notion of Cauchy sequences. Let $k < n$ be any two integers (positive or negative) and for each i satisfying $k \le i \le n$ let $0 \le a_i < p$ also be an integer. If we assume $a_k \ne 0$, then it easily follows from (1.1.6) that

$$\left| \sum_{i=k}^{n} a_i p^i \right|_p = p^{-k}. \tag{1.2.3}$$

Fix $k \in \mathbb{Z}$. An infinite sequence $\{a_k, a_{k+1}, a_{k+2}, \dots\}$, where $a_i \in \{0, 1, \dots, p-1\}$ for each $i \ge k$, and $a_k \ne 0$, determines an infinite sequence

$$\begin{aligned} x_1 &= a_k p^k \\ x_2 &= a_k p^k + a_{k+1} p^{k+1} \\ x_3 &= a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} \\ &\vdots \end{aligned}$$

of elements in $\mathbb{Q}$. By (1.2.3) it is easy to see that the sequence $x_1, x_2, x_3 \dots$ is a Cauchy sequence. Formally, we may define

$$\lim_{i \to \infty} x_i = \sum_{i=k}^{\infty} a_i p^i, \qquad (\text{with } |x_i|_p = p^{-k} \text{ for all } i = 1, 2, \dots).$$

Let $\mathbb{Z}_p$ denote the set of all elements x of the completed field $\mathbb{Q}_p$ which satisfy $|x|_p \le 1$. By (1.1.2) it easily follows that $\mathbb{Z}_p$ must be a ring with maximal ideal

$$\pi = \{x \in \mathbb{Z}_p \mid |x|_p < 1\}.$$

It is easy to check that $\pi = p \cdot \mathbb{Z}_p$. Every $x \in \mathbb{Z}_p$ can be uniquely realized as the equivalence class of a Cauchy sequence of the form

$$\{a_0, \;\; a_0 + a_1 p, \;\; a_0 + a_1 p + a_2 p^2, \;\; a_0 + a_1 p + a_2 p^2 + a_3 p^3, \quad \dots\}$$

where $0 \le a_i < p$ for $i = 0, 1, 2, \dots$ One may check this by first showing that that every element of $\mathbb{Z}_p$ contains a sequence consisting entirely of integers. Every integer may be expressed as a finite sum $a_0 + \cdots + a_N p^N$. One then shows that for the sequence to be Cauchy, the "digit" a_i must be eventually constant for each i. The ring $\mathbb{Z}_p$, can thus be realized as the set of all sums of the type:

$$\sum_{i=0}^{\infty} a_i p^i \tag{1.2.4}$$

where $0 \le a_i < p$ for each $i \ge 0$.

Suppose $x \in \mathbb{Q}_p$ does not satisfy $|x|_p \le 1$. Then we can multiply x by a suitable power p^n with $n > 0$ so that $|p^n x|_p \le 1$. It immediately follows that the field $\mathbb{Q}_p$, can thus be realized as the set of all sums of the type:

$$\sum_{i=k}^{\infty} a_i p^i \tag{1.2.5}$$

where $0 \le a_i < p$ for each $i \ge k$ and $k \in \mathbb{Z}$ arbitrary. The actual mechanics of performing addition, subtraction, multiplication, and division in the field $\mathbb{Q}_p$ is very similar to what we do in the field $\mathbb{R}$ where every element is of the form

$$a_k 10^k + a_{k-1} 10^{k-1} + \cdots \tag{1.2.6}$$

with $0 \le a_i \le 9$ for all $i \ge k$. The main difference in $\mathbb{Q}_p$ is that the expansion

$$a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} \cdots$$

goes up instead of down as in (1.2.6).

Here is an example of multiplication in $\mathbb{Q}_5$. Note that the multiplication and carrying procedures mimic the case of multiplication in $\mathbb{R}$ except that we move from left to right instead of right to left.

$$\begin{array}{rccccccccccc}
 & 2\cdot 5^{-1} & + & 4\cdot 5^{0} & + & 3\cdot 5^{1} & + & 2\cdot 5^{2} & + & \cdots & & \\
\times & 1\cdot 5^{-2} & + & 3\cdot 5^{-1} & + & 2\cdot 5^{0} & + & 1\cdot 5^{1} & + & \cdots & & \\
\hline
 & 2\cdot 5^{-3} & + & 4\cdot 5^{-2} & + & 3\cdot 5^{-1} & + & 2\cdot 5^{0} & + & \cdots & & \\
 & & + & 1\cdot 5^{-2} & + & 3\cdot 5^{-1} & + & 1\cdot 5^{0} & + & 3\cdot 5^{1} & + & \cdots \\
 & & & & + & 4\cdot 5^{-1} & + & 3\cdot 5^{0} & + & 2\cdot 5^{1} & + & \cdots \\
 & & & & & & + & 2\cdot 5^{0} & + & 4\cdot 5^{1} & + & \cdots \\
\hline
 & 2\cdot 5^{-3} & + & 0\cdot 5^{-2} & + & 1\cdot 5^{-1} & + & 0\cdot 5^{0} & + & 1\cdot 5^{1} & + & \cdots
\end{array}$$

We give one more example of the type of infinite expansion that occurs in $\mathbb{Q}_p$ which is analogous to the expansion $\frac{1}{3} = 0.33333\ldots$ that occurs in $\mathbb{R}$.

Example 1.2.7 *Let a be an integer coprime to the prime p. Let $f \ge 1$ be a fixed integer. Then there exist integers $\bar{a}, a_1, a_2, \ldots$ such that*

$$\frac{1}{a} = \bar{a} + a_f p^f + a_{f+1} p^{f+1} + a_{f+2} p^{f+2} + \cdots \in \mathbb{Q}_p$$

where $a \cdot \bar{a} \equiv 1 \pmod{p^f}$ with $0 < \bar{a} < p^f$ and $0 \le a_i < p$ for $i = f, f+1, f+2, \ldots$

Since $|a^{-1}|_p = 1$ it follows that a^{-1} must be in $\mathbb{Z}_p$ and, thus, have an expansion of type (1.2.4). We require

$$a \cdot \left(\bar{a} + a_f p^f + a_{f+1} p^{f+1} + \cdots\right) = 1$$

from which it easily follows that $a\bar{a} \equiv 1 \pmod{p^f}$.

Note that p-adic expansions of p-adic numbers are always unique. This is not the case for decimal expansions of real numbers. For example: $1.000\ldots = 0.999\ldots$

Algebraic construction of $\mathbb{Q}_p$: Let $A_1, A_2, A_3, \ldots$ be an infinite set of groups, rings, or fields. We assume that for every pair of positive integers i, j with $i > j$ there exists a homomorphism

$$f_{i,j} : A_i \to A_j. \tag{1.2.8}$$

Assume also that whenever i, j, k are positive integers satisfying $i > j > k$, that

$$f_{i,k} = f_{j,k} \circ f_{i,j}. \tag{1.2.9}$$

Definition 1.2.10 (Inverse limit) *Let $A_1, A_2, A_3, \ldots$ be an infinite set of groups, rings, or fields. Assume that for all positive integers $i > j$ that homomorphisms $f_{i,j}$ exist satisfying (1.2.8), (1.2.9). Then the inverse limit of the A_i, denoted*

$$\varprojlim A_i$$

is defined to be the set of all infinite sequences $(a_1, a_2, a_3, \ldots)$ where $a_i \in A_i$ for all $i \geq 1$ and $f_{i,j}(a_i) = a_j$ for all $i > j \geq 1$.

The inverse limit inherits the algebraic structure of the sets A_i. It will be either a group, ring or field.

In the algebraic approach to the construction of $\mathbb{Q}_p$ we first construct (using the inverse limit) the ring of p-adic integers, denoted $\mathbb{Z}_p$. The field $\mathbb{Q}_p$ is then constructed as the field of quotients of $\mathbb{Z}_p$, consisting of all elements of the form a/b with $a, b \in \mathbb{Z}_p$ and $b \neq 0$. Note that $\mathbb{Z}_p$ is an integral domain.

Let p be a prime and let i be a positive integer. Then the set

$$A_i := \{a_0 + a_1 p + \cdots a_{i-1} p^{i-1} \mid 0 \leq a_\ell < p \text{ for all } 0 \leq \ell < i\} \tag{1.2.11}$$

determines a finite ring with p^i elements which is canonically identified with the quotient ring $(\mathbb{Z}/p^i\mathbb{Z})$. The algebraic operations are addition and multiplication modulo p^i. For every $i > j$, we have the canonical homorphism $f_{i,j} : A_i \to A_j$ defined by

$$f_{i,j}\left(a_0 + a_1 p + \cdots a_{i-1} p^{i-1}\right) = a_0 + a_1 p + \cdots a_{j-1} p^{j-1},$$

which simply drops off the tail end terms in the sum. It easily follows from definition 1.2.10 that an element of the inverse limit is a sequence of the form

$$(a_0, \;\; a_0 + a_1 p, \;\; a_0 + a_1 p + a_2 p^2, \;\; a_0 + a_1 p + a_2 p^2 + a_3 p^3, \quad \ldots).$$

Formally, we define the infinite sum $\sum_{i=0}^{\infty} a_i p^i$ to be the sequence above. Then

$$\lim_{\longleftarrow} \left(\mathbb{Z}/p^i\mathbb{Z}\right) = \left\{ \sum_{i=0}^{\infty} a_i p^i \;\middle|\; 0 \le a_i < p \text{ for all } i \ge 0 \right\}. \tag{1.2.12}$$

Definition 1.2.13 (ring of p-adic integers $\mathbb{Z}_p$) *Let p be a prime number. The ring of p-adic integers $\mathbb{Z}_p$ is defined to be the inverse limit of finite rings given by (1.2.11).*

§1.3 Adeles and ideles over $\mathbb{Q}$

The completion of $\mathbb{Q}$ with respect to the archimedean absolute value $|\,|_\infty$ is just $\mathbb{R}$ which we also denote as $\mathbb{Q}_\infty$. Formally, the ring of adeles over $\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}$, is a ring determined by the restricted product (relative to the subgroups $\mathbb{Z}_p$)

$$\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod_p \mathbb{Q}_p,$$

where restricted product (relative to the subgroups $\mathbb{Z}_p$) means that all but finitely many of the components in the product are in $\mathbb{Z}_p$.

Definition 1.3.1 (Adeles) *The ring of adeles over $\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}$, is defined by*

$$\mathbb{A}_\mathbb{Q} := \left\{ \{x_\infty, x_2, x_3, \ \ldots\} \;\middle|\; x_v \in \mathbb{Q}_v \text{ ($\forall\ v \le \infty$), } x_p \in \mathbb{Z}_p \text{ ($\forall$ but finitely many p)} \right\}.$$

Given two adeles

$$x = \{x_\infty,\ x_2,\ x_3, \quad \ldots\}, \qquad x' = \{x'_\infty,\ x'_2,\ x'_3, \quad \ldots\},$$

we define addition and multiplication (the ring operations) as follows

$$x + x' := \{x_\infty + x'_\infty,\ x_2 + x'_2,\ x_3 + x'_3, \quad \ldots\}$$
$$x \cdot x' := \{x_\infty \cdot x'_\infty,\ \ x_2 \cdot x'_2,\ \ x_3 \cdot x'_3, \quad \ldots\}.$$

Recall that a topological space X is called locally compact if every point of X has a compact neighborhood. For example, $\mathbb{Q}_p$ is locally compact and $\mathbb{Z}_p$ is compact. Furthermore, $\mathbb{A}_\mathbb{Q}$ can be made into a locally compact topological ring by taking as a basis for the topology all sets of the form

$$U \times \prod_{p \notin S} \mathbb{Z}_p$$

where S is any finite set of primes containing ∞, and U is any open subset in the product topology on the finite product $\prod_{v \in S} \mathbb{Q}_v$.

The ideles of $\mathbb{Q}$ are defined to be the multiplicative subgroup of $\mathbb{A}_\mathbb{Q}$, denoted $\mathbb{A}_\mathbb{Q}^\times$.

Definition 1.3.2 (Ideles) *The multiplicative group of ideles over $\mathbb{Q}$, denoted $\mathbb{A}_{\mathbb{Q}}^{\times}$, is defined by*

$$\mathbb{A}_{\mathbb{Q}}^{\times} := \Big\{ \{x_\infty, x_2, \ \ldots\} \in \mathbb{A}_{\mathbb{Q}} \ \Big| \ x_v \in \mathbb{Q}_v^{\times} \ (\forall v), \ x_p \in \mathbb{Z}_p^{\times} \ (\forall \text{ but finitely many } p) \Big\}.$$

Here $\mathbb{Z}_p^{\times}$ denotes the multiplicative group of units of $\mathbb{Z}_p$. Clearly, $u \in \mathbb{Z}_p^{\times}$ if and only if $|u|_p = 1$. The ideles over $\mathbb{Q}$ also form a locally compact topological group with the basis of the topology consisting of the open sets

$$U \times \prod_{p \notin S} \mathbb{Z}_p^{\times}$$

where U is an open set in $\prod_{v \in S} \mathbb{Q}_v^{\times}$ and S is any finite set of primes containing ∞. Here, the topology on the finite product $\prod_{v \in S} Q_v^{\times}$ is the product topology.

Warning: The topology of the ideles is not the topology induced from the adeles. It is quite different.

Definition 1.3.3 (Finite adeles) *The ring of finite adeles over $\mathbb{Q}$, denoted $\mathbb{A}_{\text{finite}}$, is defined by*

$$\mathbb{A}_{\text{finite}} := \Big\{ \{x_2, x_3, \ \ldots\} \ \Big| \ x_p \in \mathbb{Q}_p \ (\forall \ p < \infty), \ x_p \in \mathbb{Z}_p \ (\forall \text{ but finitely many } p) \Big\}.$$

There is a natural embedding of $\mathbb{A}_{\text{finite}}$ into $\mathbb{A}_{\mathbb{Q}}$ given by

$$\{x_2, x_3, \ \ldots\} \mapsto \{0, x_2, x_3, \ \ldots\}.$$

Definition 1.3.4 (Finite ideles) *The group of finite ideles over $\mathbb{Q}$, denoted $\mathbb{A}_{\text{finite}}^{\times}$, is defined by*

$$\mathbb{A}_{\text{finite}}^{\times} := \Big\{ \{x_2, x_3, \ \ldots\} \ \Big| \ x_p \in \mathbb{Q}_p^{\times} \ (\forall \ p < \infty), \ x_p \in \mathbb{Z}_p^{\times} \ (\forall \text{ but finitely many } p) \Big\}.$$

There is a natural embedding of $\mathbb{A}_{\text{finite}}^{\times}$ into $\mathbb{A}_{\mathbb{Q}}^{\times}$ given by

$$\{x_2, x_3, \ \ldots\} \mapsto \{1, x_2, x_3, \ \ldots\}.$$

§1.4 Action of $\mathbb{Q}$ on the adeles and ideles

The ring $\mathbb{Q}$ can be embedded in the adeles as follows. It is clear that for any fixed $q \in \mathbb{Q}$ that $|q|_v > 1$ for only finitely many $v \le \infty$. Thus q lies in $\mathbb{Z}_p$ for all but finitely many $p < \infty$.

Let $q \in \mathbb{Q}$. Then

$$\{q, q, q, \ \ldots\} \in \mathbb{A}_{\mathbb{Q}}.$$

This is usually referred to as a diagonal embedding. It follows that $\mathbb{Q}$ may be considered as a subring of $\mathbb{A}_{\mathbb{Q}}$. Viewing $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{Q}$ as additive groups, it is then natural to take the quotient $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$. Another way to view this quotient is to define an additive action (denoted $+$) of $\mathbb{Q}$ on $\mathbb{A}_\mathbb{Q}$ by the formula

$$q \ + \ x := \{q + x_\infty, \ q + x_2, \ q + x_3, \ \ \dots\}$$

for all $x = \{x_\infty, x_2, x_3, \dots\} \in \mathbb{A}_\mathbb{Q}$ and all $q \in \mathbb{Q}$. Here $q + x_v$ denotes addition in $\mathbb{Q}_v$. This is a continuous action and $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}_\mathbb{Q}$ in the sense that for each $q \in \mathbb{Q}$, there is a subset $U \subset \mathbb{A}_\mathbb{Q}$, which is open in the topology on $\mathbb{A}_\mathbb{Q}$, such that $U \cap \mathbb{Q} = \emptyset$.

We now introduce the notion of a fundamental domain for the action of an arbitrary group on an arbitrary set X.

Definition 1.4.1 (Fundamental domain) *Let a group G act on a set X (on the left). A fundamental domain for this action is a subset $D \subset X$ which satisfies the following two properties:*

(1) *For each $x \in X$, there exists $d \in D$ and $g \in G$ such that $gx = d$.*

(2) *The choice of d in (1) is unique.*

Remarks: A fundamental domain is precisely a choice of one point from each orbit of G. If $G\backslash X$ is the quotient space with the quotient topology and $\pi : X \to G\backslash X$ is the quotient map, then the fundamental domain is the image of a section $\sigma : G\backslash X \to X$. (This is a set theoretic section, it need not be continuous).

The construction of an explicit fundamental domain for the action of the additive group $\mathbb{Q}$ on the adele group $\mathbb{A}_\mathbb{Q}$ is equivalent to a generalization of the ancient Chinese remainder theorem.

Theorem 1.4.2 (Chinese Remainder Theorem) *Let $p_1, p_2, \dots p_n$ be distinct primes. Let $e_1, e_2, \dots, e_n$ be positive integers and $c_1, c_2, \dots, c_n$ be arbitrary integers. Then the system of linear congruences*

$$\begin{aligned} x &\equiv c_1 \pmod{p_1^{e_1}} \\ x &\equiv c_2 \pmod{p_2^{e_2}} \\ &\ \ \vdots \\ x &\equiv c_n \pmod{p_n^{e_n}} \end{aligned}$$

has a unique solution $x \pmod{p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}}$.

Proof: A simple proof can be obtained by explicitly constructing a solution to the system of linear congruences. Set $N = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. For each $1 \le i \le n$ define an integer u_i by the condition

$$\frac{N}{p_i^{e_i}} \cdot u_i \ \equiv \ 1 \pmod{p_i^{e_i}}.$$

Then one easily checks that the element

$$x \ \equiv \ c_1 \frac{N}{p_1^{e_1}} \cdot u_1 + c_2 \frac{N}{p_2^{e_2}} \cdot u_2 + \quad \cdots \quad + c_n \frac{N}{p_n^{e_n}} \cdot u_n$$

satisfies $x \equiv c_i \pmod{p_i^{e_i}}$ for all $1 \le i \le n$. We leave the proof of uniqueness to the reader.

□

Example 1.4.3 Consider the system of linear congruences

$$x \equiv 2 \pmod{3^2}$$
$$x \equiv 1 \pmod{5^3}$$
$$x \equiv 3 \pmod{7}.$$

Then u_1 is defined by the congruence $5^3 \cdot 7 \cdot u_1 \equiv 1 \pmod{3^2}$, and $u_1 = 5$. Similarly, $3^2 \cdot 7 \cdot u_2 \equiv 1 \pmod{5^3}$ and $u_2 = 2$, while $3^2 \cdot 5^3 \cdot u_3 \equiv 1 \pmod{7}$ and $u_3 = 3$. It follows that

$$x \;\equiv\; 2 \cdot 5^3 \cdot 7 \cdot 5 + 3^2 \cdot 7 \cdot 2 + 3 \cdot 3^2 \cdot 5^3 \cdot 3 \;\equiv\; 3251 \pmod{3^2 \cdot 5^3 \cdot 7}.$$

A modern version of the Chinese remainder theorem (1.4.2) can be given in terms of p-adic absolute values.

Theorem 1.4.4 (Weak approximation) *Let $p_1, p_2, \ldots, p_n$ be distinct primes. Let $c_i \in \mathbb{Q}_{p_i}$ for each $i = 1, 2, \ldots, n$. Then for every $\epsilon > 0$, there exists an $\alpha \in \mathbb{Q}$ such that*

$$|\alpha - c_i|_{p_i} < \epsilon$$

for all $1 \le i \le n$. Furthermore, α may be chosen so that the denominator, when written in lowest terms, is not divisible by any primes other than $p_1, \ldots, p_n$.

Proof: The general case follows easily from the case when $c_i \in \mathbb{Z}_{p_i}$ for all i. As $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we may then replace c_i by $c_i' \in \mathbb{Z}$. At this point the statement reduces to the classical form, given in Theorem 1.4.2. □

Proposition 1.4.5 (Strong approximation for adeles) *A fundamental domain D for $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ is given by*

$$D = \Big\{ \{x_\infty, x_2, x_3, \; \ldots\} \;\Big|\; 0 \le x_\infty < 1, \; x_p \in \mathbb{Z}_p \text{ for all finite primes } p \Big\}$$
$$= [0,1) \cdot \prod_p \mathbb{Z}_p.$$

That is, we have

$$\mathbb{A}_{\mathbb{Q}} = \bigcup_{\beta \in \mathbb{Q}} \{\beta + D\}, \qquad (\textit{disjoint union}).$$

Proof: Following definition 1.4.1, it is enough to show that every element in $\mathbb{A}_{\mathbb{Q}}$ can be uniquely expressed as $d + q$ for $d \in D$ and $q \in \mathbb{Q}$.

Fix $x = \{x_\infty, x_2, x_3, \ldots,\} \in \mathbb{A}_{\mathbb{Q}}$. Apply Theorem 1.4.4 with $p_1, \ldots, p_n$ being the finite set of primes such that $x_{p_i} \notin \mathbb{Z}_{p_i}$, $c_i = x_{p_i}$ and $\epsilon = 1$. We obtain $\alpha \in \mathbb{Q}$ such that $|x_p - \alpha|_p \le 1$ for all p. For $t \in \mathbb{R}$, let $[t]$ denote the greatest integer not exceeding t. Since $\alpha \in \mathbb{Z}_p$ for all $p \ne p_i$, it follows that $x_p - \alpha - [x_\infty - \alpha] \in \mathbb{Z}_p$

for all finite primes p and $x_\infty - \alpha - [x_\infty - \alpha] \in [0,1)$. We have thus found $q = -\alpha - [x_\infty - \alpha] \in \mathbb{Q}$ and $d \in D$ such that

$$x + \{q, q, q, \ldots\} = d.$$

Next, we consider uniqueness. Suppose there exists $q' \in \mathbb{Q}$ and $d' \in D$ such that $x + \{q', q', q', \ldots\} = d'$. This implies $\{q,q,q,\ldots\} - \{q',q',q',\ldots\} = d - d'$. But then $q - q'$ is an integer at all finite places and at ∞ we must have $-1 < q - q' < 1$. It immediately follows that $q = q'$.

Finally, the proof that the union of all rational translates of the fundamental domain D gives $\mathbb{A}_{\mathbb{Q}}$ follows immediately from definition 1.4.1 of a fundamental domain. □

Next, we consider the multiplicative action of $\mathbb{Q}^\times$ on the ideles $\mathbb{A}_{\mathbb{Q}}^\times$ which we denote by $\cdot$ which is defined by

$$q \cdot x = \{q \cdot x_\infty,\ q \cdot x_2,\ q \cdot x_3,\ \ldots\}$$

for all $x = \{x_\infty, x_2, x_3, \ldots\} \in \mathbb{A}_{\mathbb{Q}}^\times$ and $q \in \mathbb{Q}^\times$. Here $q \cdot x_v$ denotes multiplication in $\mathbb{Q}_v$.

Proposition 1.4.6 (Strong approximation for ideles) *A fundamental domain D for $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ is given by*

$$\begin{aligned} D &= \left\{ \{x_\infty, x_2, x_3, \ \ldots\} \;\middle|\; 0 \le x_\infty < \infty,\ x_p \in \mathbb{Z}_p^\times \text{ for all finite primes } p \right\} \\ &= (0,\infty) \cdot \prod_p \mathbb{Z}_p^\times. \end{aligned}$$

That is, we have

$$\mathbb{A}_{\mathbb{Q}}^\times = \bigcup_{\alpha \in \mathbb{Q}^\times} \alpha \cdot D. \qquad (\textit{disjoint union.})$$

Proof: Following definition 1.4.1, it is enough to show that every element in $\mathbb{A}_{\mathbb{Q}}^\times$ can be uniquely expressed as $d \cdot q$ for some $d \in D$ and some $q \in \mathbb{Q}^\times$.

The proof is very similar to the proof of proposition 1.4.5 and is left to the reader. □

§1.5 p-adic integration

Let us consider complex valued continuous functions f defined on $\mathbb{Q}_p$. We would like to define the notion of the integral of f, denoted $\int_A f(x)\,dx$, taken over a subset $A \subset \mathbb{Q}_p$.

Definition 1.5.1 (Finitely additive measure) *A finitely additive measure μ on $\mathbb{Q}_p$ is a map from the set of compact subsets of $\mathbb{Q}_p$ to the non-negative real numbers which satisfies*

$$\mu\big(U_1 \cup U_2 \cup \cdots \cup U_n\big) = \mu(U_1) + \cdots + \mu(U_n),$$

for all compact subsets $U_1, \ldots, U_n$ of $\mathbb{Q}_p$ which are pairwise disjoint.

It is easy to see that a finitely additive measure μ on $\mathbb{Q}_p$ must satisfy

$$\mu(a + p^n\mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu\left(a + bp^n + p^{n+1}\mathbb{Z}_p\right)$$

for all $0 \le a \le p-1$ and $n \in \mathbb{Z}$, because the compact set on the left side is the disjoint union of the ones on the right.

Definition 1.5.2 (Locally constant function) *A function $f : \mathbb{Q}_p \to \mathbb{C}$ is said to be locally constant on a subset $V \subset \mathbb{Q}_p$ if for every $x \in V$ there exists an open set $U \subset V$ containing x such that $f(x) = f(u)$ for all $u \in U$. The function f is said to be locally constant if it is locally constant on all of $\mathbb{Q}_p$.*

Note that any locally constant function $f : \mathbb{Q}_p \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form

$$f(x) = \sum_{i=1}^{\infty} c_i \cdot 1_{U_i}(x)$$

where $c_i \in \mathbb{C}$ and 1_{U_i} is the characteristic function of U_i. Here U_i are open subsets of $\mathbb{Q}_p$ for $i = 1, 2, \ldots$ The locally constant functions are the analogue of step functions in the classical integration theory on $\mathbb{R}$.

Definition 1.5.3 (Integration of locally constant, compactly supported functions on $\mathbb{Q}_p$) *Let $f : \mathbb{Q}_p \to \mathbb{C}$ be a locally constant function. Let μ be a finitely additive measure on $\mathbb{Q}_p$ as in definition 1.5.1 and assume*

$$A = U_1 \cup U_2 \cup \cdots \cup U_n$$

is a disjoint union of compact open sets $U_i \subset \mathbb{Q}_p$ such that f is the constant function $c_i \in \mathbb{C}$ on each U_i for $i = 1, 2, \ldots, n$. Then we define

$$\int_A f(x)\, d_\mu(x) = c_1\mu(U_1) + c_2\mu(U_2) + \cdots + c_n\mu(U_n).$$

Remarks: (1) We shall refer to $d_\mu(x)$ as the "differential induced from" the finitely additive measure μ. This is perhaps best thought of as a common and extremely useful abuse of notation. What has been defined rigorously is a linear functional on the space of locally constant, compactly supported functions f. We shall be considering other, similar, functionals, coming from closely-related finitely additive measures. The descriptions are easiest to grasp when presented as relationships among the "differentials." The interested reader should have no difficulty recovering the definition in terms of finitely additive measures.

(2) It is possible, in a very straightforward manner, to extend the definition of the integral given in 1.5.3 to an infinite disjoint union $A = \bigcup_{i=1}^{\infty} U_i$ provided the sum $\sum_{i=1}^{\infty} c_i\mu(U_i)$ converges absolutely. Furthermore, one may extend a finitely additive

measure as above to a measure on a σ-algebra of sets, define associated classes of measurable and integrable functions, etc. See, [Halmos, 1950], [Hewitt-Ross, 1979], [Bourbaki, 2004]. We shall not need to do this, however, since the construction of the standard automorphic L-functions, which is the main theme of this book, only requires the integration of locally constant functions.

(3) Because the compact open sets $a + p^n\mathbb{Z}_p$ form a basis of open sets for $\mathbb{Q}_p$, we are able to integrate any locally constant, compactly supported function using only the values of μ on these sets.

Example 1.5.4 (Haar measure on $\mathbb{Q}_p$) *Let $a, n \in \mathbb{Z}$ with $0 \le a \le p-1$. We define*

$$\mu_{\text{Haar}}\,(a + p^n\mathbb{Z}_p) = p^{-n}.$$

We also set $d_{\mu_{\text{Haar}}}(x) = dx$. Haar measure is (obviously) invariant under additive translations. Note that μ_{Haar} can be arbitrarily large on the compact sets $a + p^n\mathbb{Z}_p$ when $n \to -\infty$.

We now give an example of a simple p-adic integral. Let s be a complex number with $\Re(s) > -1$. Then the function $|x|_p^s$ is a locally constant function on $\mathbb{Q}_p - \{0\}$. We compute the integral of $|x|_p^s$ over $\mathbb{Z}_p - \{0\}$, the non-zero p-adic integers. Let

$$\mathbb{Z}_p^\times = (1 + p\mathbb{Z}_p) \cup (2 + p\mathbb{Z}_p) \cup \cdots \cup (p - 1 + p\mathbb{Z}_p)$$

denote the units (invertible elements) in $\mathbb{Z}_p$, which are characterized by the fact that $u \in \mathbb{Z}_p^\times$ if and only if $|u|_p = 1$. Clearly $\mathbb{Z}_p - \{0\} = \bigcup\limits_{n=0}^{\infty} p^n\mathbb{Z}_p^\times$ is a disjoint union and $\mu_{\text{Haar}}\left(\mathbb{Z}_p^\times\right) = \frac{p-1}{p}$.

Example 1.5.5 *Let dx be the differential induced from the Haar measure as in example 1.5.4. We have*

$$\int\limits_{\mathbb{Z}_p - \{0\}} |x|_p^s\, dx \;=\; \sum_{n=0}^{\infty} \int\limits_{p^n\mathbb{Z}_p^\times} |x|_p^s\, dx \;=\; \frac{p-1}{p} \sum_{n=0}^{\infty} p^{-n} \cdot p^{-ns} \;=\; \frac{p-1}{p(1 - p^{-1-s})}.$$

In example 1.5.5 we have reduced the integral over $\mathbb{Z}_p^\times - \{0\}$ to an infinite sum of integrals over compact sets which can be computed as in definition 1.5.3. The condition $\Re(s) > -1$ ensures that the above infinite sum converges absolutely. We also note that $\int\limits_{\mathbb{Z}_p - \{0\}} |x|_p^s\, dx = \int\limits_{\mathbb{Z}_p} |x|_p^s\, dx$ since the integral over the point $\{0\}$ is 0.

As noted above, the Haar measure given in 1.5.4 is invariant by additive changes of variable. When we make a multiplicative change of variables, we get

$$\int_{\mathbb{Q}_p} f(ax)\, dx = |a|_p \int_{\mathbb{Q}_p} f(x)\, dx. \tag{1.5.6}$$

Definition 1.5.7 (Multiplicative Haar measure on $\mathbb{Q}_p^\times$) *Let dx be as in example 1.5.4. For $x \in \mathbb{Q}_p^\times = \mathbb{Q}_p - \{0\}$, we define*

$$d^\times x = \frac{p}{p-1}\,\frac{dx}{|x|_p} = \left(1 - p^{-1}\right)^{-1}\,\frac{dx}{|x|_p}.$$

The differential $d^\times x$ satisfies the following two important properties. First of all, it is invariant under transformations $x \to yx$ for any fixed $y \in \mathbb{Q}_p^\times$. That is, for any locally constant function f such that the integral

$$\int_{\mathbb{Q}_p^\times} f(x) d^\times x$$

converges, and for any $y \in \mathbb{Q}_p^\times$, we have

$$\int_{\mathbb{Q}_p^\times} f(xy) d^\times x = \int_{\mathbb{Q}_p^\times} f(x) d^\times x,$$

for any $y \in \mathbb{Q}_p^\times$. (The general case reduces to the special case when the function f is the characteristic function of an open ball, which is a straightforward exercise.) Thus, $d^\times x$ is invariant under multiplication, which is why it is called a multiplicative Haar measure. Secondly, it satisfies $\int\limits_{\mathbb{Z}_p^\times} d^\times x = 1$.

Example 1.5.8 *Let $d^\times x$ be as in definition 1.5.7. Then for $\Re(s) > 0$, we have*

$$\int\limits_{\mathbb{Z}_p - \{0\}} |x|_p^s \, d^\times x \;=\; \left(1 - p^{-1}\right)^{-1} \sum_{n=0}^{\infty} \int\limits_{p^n \mathbb{Z}_p^\times} |x|_p^s \frac{dx}{|x|_p} \;=\; \sum_{n=0}^{\infty} p^{-ns} \;=\; \frac{1}{1 - p^{-s}}.$$

§1.6 p-adic Fourier transform

Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We shall say that f has rapid decay at ∞ if for each $m > 0$ there exists a fixed constant $C > 0$ such that

$$|x|_\infty^m \, |f(x)|_{\mathbb{C}} < C$$

for $|x|_\infty$ sufficiently large. Here $|x|_\infty$ is the ordinary absolute value on $\mathbb{R}$ as in (1.1.5), and $|\ |_{\mathbb{C}}$ is the ordinary absolute value on $\mathbb{C}$. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be Schwartz if it is smooth (infinitely differentiable) and all of its derivatives have rapid decay at infinity.

The Fourier transform of f, denoted $\hat{f}$, is defined by

$$\hat{f}(x) = \int\limits_{\mathbb{R}} f(y) e_\infty(-xy)\, dy, \tag{1.6.1}$$

where $e_\infty(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$.

Theorem 1.6.2 (Fourier inversion on $\mathbb{R}$) *Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Let $\hat{f}$ be defined by (1.6.1). Then $\hat{f}$ is again a Schwartz function and $\hat{\hat{f}}(x) = f(-x)$.*

Proof: See [Lang, 1983].

We want to generalize (1.6.1) and theorem 1.6.2 to p-adic fields. The first step required to do this is to obtain an analogue of the additive character $e_\infty(x)$ which satisfies $e_\infty(x+y) = e_\infty(x)e_\infty(y)$ for all $x, y \in \mathbb{R}$. Accordingly, we define the function $e_p : \mathbb{Q}_p \to \mathbb{C}$.

Definition 1.6.3 (Additive character on $\mathbb{Q}_p$) *Let $e_p : \mathbb{Q}_p \to \mathbb{C}$ be defined by*

$$e_p(x) = e^{-2\pi i \{x\}}$$

where

$$\{x\} = \begin{cases} \sum\limits_{i=-k}^{-1} a_i p^i, & \text{if } x = \sum\limits_{i=-k}^{\infty} a_i p^i \in \mathbb{Q}_p \text{ with } k > 0,\ 0 \le a_i \le p-1, \\ 0, & \text{otherwise.} \end{cases}$$

Remarks: We think of $\{x\}$ as the fractional part of $x \in \mathbb{Q}_p$. Clearly

$$e_p(x+y) = e_p(x) \cdot e_p(y)$$

for all $x, y \in \mathbb{Q}_p$. Note the minus sign in the definition of e_p. The minus sign plays an important role in the adelic Fourier theory. Let us mention that Tate and some other authors include a minus sign in e_∞, rather than e_p.

Lemma 1.6.4 *Let $n \in \mathbb{Z}$. Then*

$$\int\limits_{p^n\mathbb{Z}_p} e_p(x)\, dx = \begin{cases} p^{-n}, & \text{if } n \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: If $n \ge 0$ then the integrand is identically equal to 1, and hence the integral is equal to the measure of the domain of integration, which is p^{-n}.

If $n < 0$, then there exists $y \in p^n\mathbb{Z}_p$ such that $e_p(y) \ne 1$. Making the change of variables $x \to x + y$ in the integration, we obtain

$$\int\limits_{p^n\mathbb{Z}_p} e_p(x)\, dx = e_p(y) \int\limits_{p^n\mathbb{Z}_p} e_p(x)\, dx.$$

Since $e_p(y) \ne 1$, the integral must be 0.

Proposition 1.6.5 *Let $n \in \mathbb{Z}$. Then*

$$\int\limits_{p^n\mathbb{Z}_p^\times} e_p(x)\, dx = \begin{cases} p^{-n}\left(1 - p^{-1}\right), & \text{if } n \ge 0, \\ -1, & \text{if } n = -1, \\ 0, & \text{if } n < -1. \end{cases}$$

Proof: Since $\mathbb{Z}_p^\times = \mathbb{Z}_p - p\mathbb{Z}_p$ it follows, after multiplying by p^n, that we may write $p^n\mathbb{Z}_p^\times = p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p$. Consequently

$$\int\limits_{p^n\mathbb{Z}_p^\times} e_p(x)\, dx = \int\limits_{p^n\mathbb{Z}_p} e_p(x)\, dx \; - \int\limits_{p^{n+1}\mathbb{Z}_p} e_p(x)\, dx. \tag{1.6.6}$$

If $n \geq 0$ then $e_p(x) \equiv 1$ in both of the integrals on the right side of (1.6.6) so the value of the integral is given by

$$\mu_{Haar}(p^n\mathbb{Z}_p) - \mu_{Haar}(p^{n+1}\mathbb{Z}_p) = \frac{1}{p^n} - \frac{1}{p^{n+1}} = p^{-n}\left(1 - p^{-1}\right).$$

If $n = -1$, then e_p is nontrivial on $p^n\mathbb{Z}_p$ and trivial on $p^{n+1}\mathbb{Z}_p$, so the value of the integral in (1.6.6) is just $0 - \mu_{Haar}(\mathbb{Z}_p) = -1$. If $n < -1$, then since e_p is nontrivial in both of the integrals on the right side of (1.6.6) the integral is just 0 in this case.
□

Proposition 1.6.7 *Let*

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

denote the characteristic function of a subset $A \subset \mathbb{Q}_p$. *Let* $n \in \mathbb{Z}$. *Then we have*

$$\int\limits_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) \cdot e_p(-xy)\,dx = \begin{cases} p^{-n}, & \text{if } y \in p^{-n}\mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: If $y \in p^{-n}\mathbb{Z}_p$ then $xy \in \mathbb{Z}_p$ for all $x \in p^n\mathbb{Z}_p$, so the integral is just $\int_{p^n\mathbb{Z}_p} dx = p^{-n}$. If $y \notin p^{-n}\mathbb{Z}_p$ then e_p is nontrivial and the integral vanishes.
□

Theorem 1.6.8 (Fourier inversion on $\mathbb{Q}_p$) *Let* $f : \mathbb{Q}_p \to \mathbb{C}$ *be a locally constant compactly supported function as in definition 1.5.2. Let* $\hat{f}$ *be defined by*

$$\boxed{\hat{f}(x) = \int_{\mathbb{Q}_p} f(y)e_p(-xy)\,dy.}$$

Then $\hat{f}$ *is again a locally constant compactly supported function and* $\hat{\hat{f}}(x) = f(-x)$.

Proof: We first show that $\hat{f}$ is a locally constant compactly supported function. Every locally constant compactly supported function on $\mathbb{Q}_p$ can be expressed as a finite linear combination of characteristic functions of compact open sets of the form $a + p^n\mathbb{Z}_p$ with $a \in \mathbb{Q}_p$. and $n \in \mathbb{Z}$. Since integration is a linear function it suffices to show that the Fourier transform of the characteristic function $1_{a+p^n\mathbb{Z}_p}$ is again a locally constant compactly supported function. We compute

$$\begin{aligned} \widehat{1}_{a+p^n\mathbb{Z}_p}(y) = \int\limits_{\mathbb{Q}_p} 1_{a+p^n\mathbb{Z}_p}(x) \cdot e_p(-xy)\,dx &= \int\limits_{a+p^n\mathbb{Z}_p} e_p(-xy)\,dx \\ &= \int\limits_{p^n\mathbb{Z}_p} e_p\big(-(a+x)y\big)\,dx \\ &= e_p(-ay)\int\limits_{p^n\mathbb{Z}_p} e_p(-xy)\,dx \\ &= e_p(-ay)p^{-n} \cdot 1_{p^{-n}\mathbb{Z}_p}(y), \end{aligned} \tag{1.6.9}$$

where the last step in the above calculation follows from proposition 1.6.7 .

To show that $\hat{\hat{f}}(x) = f(-x)$ it suffices to check it for the case that $f = 1_{a+p^n\mathbb{Z}_p}$ is a characteristic function of the compact open set $a+p^n\mathbb{Z}_p$. It follows from (1.6.9) that for any $y \in \mathbb{Q}_p$,

$$\begin{aligned}\hat{\hat{1}}_{a+p^n\mathbb{Z}_p}(y) &= \int_{\mathbb{Q}_p} \hat{1}_{a+p^n\mathbb{Z}_p}(x)\cdot e_p(-xy)\,dx \\ &= p^{-n}\int_{\mathbb{Q}_p} 1_{p^{-n}\mathbb{Z}_p}(x)\cdot e_p(-(a+y)x)\,dx \\ &= 1_{p^n\mathbb{Z}_p}(a+y),\end{aligned}$$

where the last step follows from proposition 1.6.7 (with n replaced by $-n$). But $1_{p^n\mathbb{Z}_p}(a+y) = 1_{a+p^n\mathbb{Z}_p}(-y)$ because $-p^n\mathbb{Z}_p = p^n\mathbb{Z}_p$.
□

§1.7 Adelic Fourier transform

Recall definition 1.3.1 which states that the adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined by

$$\mathbb{A}_{\mathbb{Q}} := \Big\{ \{x_\infty, x_2, x_3, \ \ldots\} \ \Big|\ x_v \in \mathbb{Q}_v\,(\forall\ v \le \infty),\ \ x_v \in \mathbb{Z}_v\,(\forall \text{ but finitely many } v)\Big\}.$$

In order to define a Fourier transform on the global ring $\mathbb{A}_{\mathbb{Q}}$ it is first necessary to construct an appropriate additive character as in the local definition 1.6.3.

Definition 1.7.1 (Additive adelic character) *We shall define an additive adelic character $e : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ as follows. For $x = \{x_\infty, x_2, x_3, \ldots\} \in \mathbb{A}_{\mathbb{Q}}$ let*

$$e(x) = \prod_{v\le\infty} e_v(x_v)$$

with $e_p(x_p) = e^{-2\pi i\{x_p\}}$ as in (1.6.3) if $p < \infty$ and $e_\infty(x_\infty) = e^{2\pi i x_\infty}$ if $v = \infty$. Note that only finitely many of the terms in the product are not equal to 1.

Proposition 1.7.2 *The function $e(x)$ which is defined in definition 1.7.1 satisfies the following two properties:*

(1) additivity: *$e(x+y) = e(x)\cdot e(y)$ for all $x, y \in \mathbb{A}_{\mathbb{Q}}$;*

(2) periodicity: $e(x+\alpha) = e(x)$ for all $x \in \mathbb{A}_{\mathbb{Q}}$ and $\alpha \in \mathbb{Q}$.

Proof:

(1) The additivity follows from the additivity of each of the local exponentials $e_\infty, e_2, e_3, \ldots$ as explained in the remark after definition 1.6.3.

(2) Next we show the periodicity. Let

$$\alpha \ = \ \frac{a}{p_1^{f_1}p_2^{f_2}\cdots p_\ell^{f_\ell}}$$

where $a \in \mathbb{Z}$ and $p_1, p_2, \ldots, p_\ell$ are primes and $p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}$ is the prime factorization of the denominator of α. Now there exist integers $b_1, b_2, \ldots b_\ell$ such that

$$\frac{1}{p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}} \;=\; \frac{b_1}{p_1^{f_1}} + \frac{b_2}{p_2^{f_2}} \;+\; \cdots \;+\; \frac{b_\ell}{p_\ell^{f_\ell}}.$$

Note that this partial fraction decomposition easily follows from the theorem of Euclid which says that for any two non-zero integers r, s there exist integers x, y such that $rx + sy = (r, s)$, where (r, s) denotes the greatest common divisor of r and s.

It follows that $e(x + \alpha) = e(x)e(\alpha)$ and

$$e(\alpha) = e^{2\pi i\alpha} \prod_{i=1}^{\ell} e_{p_i}(\alpha) = e^{2\pi i\alpha} \prod_{i=1}^{\ell} e^{-2\pi i a b_i / p_i^{f_i}} = 1.$$

Here we have crucially used the minus sign in definition 1.6.3 and also used the fact that $e_p(x + z) = e_p(x)$ for all $x \in \mathbb{Q}$ and any p-adic integer $z \in \mathbb{Z}_p$. For example, in $\mathbb{Q}_{p_1}$, the element $\frac{b_2}{p_2^{f_2}} + \cdots + \frac{b_\ell}{p_\ell^{f_\ell}}$ is a p-adic integer (see example 1.2.7).

□

We now consider adelic functions $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$. We would like to extend the definition of Fourier transform to adelic functions and, in addition, we want to define a suitable space of adelic Schwartz functions so that the Fourier transform $\hat{f}$ of an adelic Schwartz function f is given by an absolutely convergent integral and $\hat{f}$ is again an adelic Schwartz function.

The above goals can be achieved by making the following definitions.

Definition 1.7.3 (Factorizable function) *An adelic function $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ is factorizable if there exist local functions: $f_v : \mathbb{Q}_v \to \mathbb{C}$ ($\forall v \le \infty$) where $f_p \equiv 1$ on $\mathbb{Z}_p$ for all but finitely many $p < \infty$, and where*

$$f(x) \;=\; f_\infty(x_\infty) f_2(x_2) f_3(x_3) \cdots \;=\; \prod_v f_v(x_v).$$

for all $x = \{x_\infty, x_2, x_3, \ldots\} \in \mathbb{A}_\mathbb{Q}$.

Definition 1.7.4 (Adelic Schwartz function) *An adelic function is said to be a Schwartz function if it can be expressed as a finite linear combination (with complex coefficients) of factorizable functions $f = \prod_{v \le \infty} f_v$ as in definition 1.7.3 where the f_v satisfy the following conditions:*

(1) *f_∞ is Schwartz (as defined in the beginning of §1.6);*
(2) *each f_p is a locally constant compactly supported function at all $p < \infty$;*
(3) *f_p is the characteristic function of $\mathbb{Z}_p$ for all but finitely many $p < \infty$.*

Next, we wish to define an integral on a suitable space of adelic functions.

Definition 1.7.5 (Adelic integral) *Suppose that $f = \prod_v f_v$ is a factorizable function as in definition 1.7.3, that f_∞ is an integrable function on $\mathbb{R}$, that for each p, the function f_p is the characteristic function of a compact set C_p, and that $C_p = \mathbb{Z}_p$ for all p outside of some finite set S. Then we define the adelic integral*

$$\int_{\mathbb{A}_\mathbb{Q}} f(x)\,dx = \int_\mathbb{R} f_\infty(x_\infty)\,dx_\infty \cdot \prod_{p\in S} \int_{\mathbb{Q}_p} f_p(x_p)\,dx_p.$$

We further define the adelic integral of finite or countably infinite linear combinations of factorizable functions of the same type, with disjoint supports by linearity, provided (in the infinite case) that the corresponding sum is absolutely convergent. Finally, we define

$$\int_U f(x)\,dx = \int_{\mathbb{A}_\mathbb{Q}} f(x) 1_U(x)\,dx,$$

for any subset U of $\mathbb{A}_\mathbb{Q}$ such that $f \cdot 1_U$ is integrable as defined above.

Remarks: (1) Note that for a function f as in definition 1.7.5, the p-adic integral

$$\int_{\mathbb{Q}_p} f_p(x_p)\,dx_p$$

is simply the p-adic Haar measure of the set C_p. In particular, it is equal to one for all p not in S. This means that we replace S by a larger set S', the product is the same.

(2) Suppose that f takes values in the positive reals, and that $K_i,\ i = 1, 2, 3, \ldots$ is an increasing family of compact subsets of $\mathbb{A}_\mathbb{Q}$, such that the union is all of $\mathbb{A}_\mathbb{Q}$. Then it is easily verified that

$$\int_{\mathbb{A}_\mathbb{Q}} f(x)\,dx = \sup_i \int_{K_i} f(x)\,dx.$$

To extend to complex valued f, we write

$$f = (u^+ - u^-) + i(v^+ - v^-)$$

where u^+, u^-, v^+, v^- are positive-real-valued.

Lemma 1.7.6 (Factorization of adelic integral) *Suppose that $f = \prod_v f_v$ is a factorizable function as in definition 1.7.3 f_∞ is an integrable function on $\mathbb{R}$, that for each p, the function f_p is a locally constant function as in definition 1.5.2 with a convergent p-adic integral as in definition 1.5.3, and that for almost all p, the function f_p is identically equal to 1 on $\mathbb{Z}_p$. Then*

$$\int_{\mathbb{A}_\mathbb{Q}} f(x)\,dx = \int_\mathbb{R} f_\infty(x_\infty)\,dx_\infty \cdot \lim_{N\to\infty} \prod_{p<N} \int_{\mathbb{Q}_p} f_p(x_p)\,dx_p,$$

provided the limit

$$\lim_{N\to\infty} \prod_{p<N} \int_{\mathbb{Q}_p} |f_p(x_p)|\,dx_p,$$

is convergent.

Proof: For f with positive real values, this follows from remark (2) above, together with the observation that any compact subset of $\mathbb{A}_\mathbb{Q}$ is contained in

$$\mathbb{R} \cdot \prod_{p<N} \mathbb{Q}_p \cdot \prod_{p \geq N} \mathbb{Z}_p$$

for some N. The supremum over all compact sets corresponding to one fixed N, is

$$\int_{\mathbb{R}} f_\infty(x_\infty)\, dx_\infty \cdot \prod_{p<N} \int_{\mathbb{Q}_p} f_p(x_p)\, dx_p.$$

Then taking a supremum over all N yields the limit. A general f may be split into positive and negative, real and imaginary parts. ☐

Remark: At first glance, $\prod_v \int_{\mathbb{Q}_v} f_v(x_v)\, dx_v$ looks like it ought to be the integral of f over the full infinite cartesian product of the fields $\mathbb{Q}_v$– a space which is properly larger than $\mathbb{A}_\mathbb{Q}$. However, one may see that this is indeed the correct definition of an integral over $\mathbb{A}_\mathbb{Q}$ by reasoning as follows. Suppose that f is a positive function. Then its integral over all of $\mathbb{A}_\mathbb{Q}$ will be the supremum of the integrals over all compact subsets of $\mathbb{A}_\mathbb{Q}$. If we restrict x to any *fixed* compact subset of $\mathbb{A}_\mathbb{Q}$, then x_p is restricted to $\mathbb{Z}_p$ for all p outside some finite set S. However, this set S *depends on the compact set*. Thus taking a supremum over all compact sets of $\mathbb{A}_\mathbb{Q}$ is equivalent to taking a supremum over compact sets of $\mathbb{Q}_v$ for each v *and* taking a supremum over the finite set S, and does indeed yield the full infinite product of local integrals.

Definition 1.7.7 (Adelic Fourier transform) *Let $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ be a factorizable adelic Schwartz function as in definition 1.7.4. Let $e : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ be given as in definition 1.7.1. Then we define the Fourier transform $\hat{f}$ by the formula*

$$\hat{f}(x) = \prod_v \int_{\mathbb{Q}_v} f_v(y_v) e_v(-x_v y_v)\, dy_v.$$

This definition may be extended to arbitrary adelic Schwartz functions by linearity.

Note that by Proposition 1.6.7 and Definition 1.7.4 (3), the integral

$$\int_{\mathbb{Q}_v} f_v(y_v) e_v(-x_v y_v)\, dy_v$$

has the value 1 for all but finitely many v so the infinite product above is well defined. If we let $dy = \prod_v dy_v$, then we may think of dy as a differential on the adeles and we may succinctly write

$$\hat{f}(x) = \int_{\mathbb{A}_\mathbb{Q}} f(y) e(-xy)\, dy.$$

Theorem 1.7.8. (Fourier inversion on the adeles) *Let $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ be a Schwartz function as in definition 1.7.4. Let $\hat{f}$ denote the Fourier transform as in definition 1.7.7. Then $\hat{f}$ is again a Schwartz function and $\hat{\hat{f}}(x) = f(-x)$.*

Proof: This follows immediately from definitions 1.7.4, 1.7.7 and theorems 1.6.2 and 1.6.8 . □

§1.8 Fourier expansion of periodic adelic functions

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be periodic if $f(x+n) = f(x)$ for all integers n. We want to generalize this notion to the adele group and develop a Fourier theory on the adele group.

Definition 1.8.1 (Periodic adelic function) *Let $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be a complex valued adelic function. The function f is said to be periodic if*

$$f(x+\alpha) = f(x)$$

for all $x \in \mathbb{A}_{\mathbb{Q}}$ and all $\alpha \in \mathbb{Q}$.

We have shown in proposition 1.7.2 that the additive adelic character $e : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ given in definition 1.7.1 is periodic. The Fourier theory of locally compact groups tells us that any periodic adelic function (satisfying certain smoothness hypotheses) can be represented as an infinite linear combination of the form

$$\sum_{\alpha \in \mathbb{Q}} b_\alpha e(\alpha x) \tag{1.8.2}$$

with $b_\alpha \in \mathbb{C}$. We shall present here a short simple proof first shown to the first author by Jacquet [Anshel-Goldfeld, 1996], (see also [Garrett, 1990]).

A natural way to construct periodic adelic functions is to take all translates by elements in $\mathbb{Q}$ of a given adelic Schwartz function.

Proposition 1.8.3 (Periodized Schwartz function) *Let $h : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be an adelic Schwartz function as in definition 1.7.4. Then the sum*

$$\sum_{\alpha \in \mathbb{Q}} h(x+\alpha), \qquad (x \in \mathbb{A}_{\mathbb{Q}}),$$

converges absolutely and uniformly on compact subsets of $\mathbb{A}_{\mathbb{Q}}$ to a periodic adelic function f which is termed a periodized Schwartz function.

Proof: It is enough to prove the theorem for Schwartz functions h which are factorizable as in definition 1.7.3. Following definition 1.7.4, we may represent

$$h(x) = \prod_{v \le \infty} h_v(x_v)$$

where $x = \{x_\infty, x_2, x_3, \dots\}$ and h_p is the characteristic function of $\mathbb{Z}_p$ for all but finitely many $p < \infty$. Fix $x = \{x_\infty, x_2, x_3, \dots\} \in \mathbb{A}_{\mathbb{Q}}$. Let $S = \{\infty, p_1, p_2, \dots, p_\ell\}$ denote the finite set of primes such that $x_p \in \mathbb{Z}_p$ and h_p is the characteristic function of $\mathbb{Z}_p$ for $p \notin S$. If $\alpha \in \mathbb{Q}$, it follows that $h_p(x_p + \alpha) = 0$ for $p \notin S$ unless α is an integer in $\mathbb{Z}_p$. Since h_v is a locally constant compactly supported function for

$v \in S$ this implies that there exists a rational integer M so that $h(x+\alpha) = 0$ unless $\alpha = \frac{n}{M}$ with $n \in \mathbb{Z}$. Therefore (at least formally)

$$\sum_{\alpha\in\mathbb{Q}} h(x+\alpha) = \sum_{n\in\mathbb{Z}} h\left(x + \frac{n}{M}\right).$$

Finally, for fixed $x \in \mathbb{A}_{\mathbb{Q}}$ we must have that $h\left(x + \frac{n}{M}\right)$ has rapid decay in n as $n \to \pm\infty$. This is because h_∞ is a classical Schwartz function and for the finitely many primes $v \in S$ the function h_v is absolutely bounded. In all other cases h_v is either 1 or 0. The stated uniformity can be obtained because M can be chosen independent of x, for x in a compact set.

□

Definition 1.8.4 (Smooth adelic function) *An adelic function $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ is said to be smooth if for any point $x_0 \in \mathbb{A}_{\mathbb{Q}}$, there exists an open set U (containing x_0) and a smooth function $f_\infty^U : \mathbb{R} \to \mathbb{C}$ such that $f(x) = f_\infty^U(x_\infty)$ for all adeles $x = \{x_\infty, x_2, \dots\} \in U$.*

Remark: Using the fact that $\mathbb{A}_{\mathbb{Q}}$ is the union of a countable increasing family of compact sets, it is not difficult to show that any smooth adelic function is a countable linear combination of functions of the type considered in definition 1.7.5, with disjoint supports. In particular, the adelic integral of a smooth adelic function is defined, provided the relevant infinite sum is convergent.

We now show that every smooth periodic adelic function can, in fact, be realized as a periodized Schwartz function.

Proposition 1.8.5 (Smooth + periodic $\implies$ periodized Schwartz) *Let $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be smooth as in definition 1.8.4. Assume that*

$$f(x+\alpha) = f(x), \quad \forall\, \alpha \in \mathbb{Q},\ x \in \mathbb{A}_{\mathbb{Q}}.$$

Then there exists an adelic Schwartz function $h : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$, (as in definition 1.7.4) such that

$$f(x) = \sum_{\alpha\in\mathbb{Q}} h(x+\alpha).$$

Proof: Assume that there exists an adelic Schwartz function $h_0 : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ such that

$$\sum_{\alpha\in\mathbb{Q}} h_0(x+\alpha) = 1, \qquad (\forall\, x \in \mathbb{A}_{\mathbb{Q}}). \tag{1.8.6}$$

Then, if $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ is any smooth periodic adelic function, we see that we may define

$$h(x) := h_0(x) f(x).$$

Consequently, proposition 1.8.5 immediately follows if we can construct a Schwartz function h_0 so that (1.8.6) holds, and, in addition, we could show that the function $h(x) = h_0(x)f(x)$ was an adelic Schwartz function as in definition 1.7.4.

To get h_0, we first need a Schwartz function $h_{0,\infty} : \mathbb{R} \to \mathbb{R}$ satisfying

$$\sum_{n=-\infty}^{\infty} h_{0,\infty}(x+n) = 1, \qquad \forall\, x \in \mathbb{R}. \tag{1.8.7}$$

Such a function may be constructed, for example, by letting $g(x)$ be any smooth function such that $supp(g) = [-\frac{1}{4}, \infty)$ and $g(x) = 1$ for all $x \geq 0$. Then put $h_{0,\infty}(x) = g(x) - g(x-1)$.

Now, for $x = \{x_\infty, \dots, x_p, \dots\} \in \mathbb{A}_{\mathbb{Q}}$ define $h_0(x) = h_{0,\infty}(x_\infty) \cdot \prod_p 1_{\mathbb{Z}_p}(x_p)$ where $1_{\mathbb{Z}_p}$ denotes the characteristic function of $\mathbb{Z}_p$. We need to show that h_0 satisfies (1.8.6), for any $x \in \mathbb{R} \times \prod_p \mathbb{Z}_p$. Any other x may be written as $\beta + y$ for $y \in \mathbb{R} \times \prod_p \mathbb{Z}_p$ and $\beta \in \mathbb{Q}$, so one has only to make a change of variables in the summation. But this reduces to (1.8.7) and we are done provided we can show that $h(x) = h_0(x) f(x)$ is an adelic Schwartz function as in definition 1.7.4. The proof that $h(x)$ is an adelic Schwartz function will follow directly from lemma 1.8.8 below.

□

Lemma 1.8.8 *Every compactly supported smooth adelic function as in definition 1.8.4 is an adelic Schwartz function as in definition 1.7.4.*

Proof: Let $h : \mathbb{A} \to \mathbb{C}$ be smooth and compactly supported. We first prove a refinement of smoothness. Recall the definition 1.3.3 of the finite adeles $\mathbb{A}_{\text{finite}} = \Big\{ \{x_2, x_3, \dots\} \;\Big|\; \{0, x_2, x_3 \dots\} \in \mathbb{A}_{\mathbb{Q}} \Big\}$.

Claim: *For all $x_{\text{finite}} \in \mathbb{A}_{\text{finite}}$ there exists an open set U_{finite} of $\mathbb{A}_{\text{finite}}$ containing x_{finite} and a smooth function $h_\infty : \mathbb{R} \to \mathbb{C}$ such that $h(y) = h_\infty(y_\infty)$ for every $y = \{y_\infty, y_{\text{finite}}\}$ in $\mathbb{R} \times U_{\text{finite}}$. Furthermore, U_{finite} may be assumed to be of the form $\prod_p U_p$ where $U_p = \mathbb{Z}_p$ for almost all p, and is of the form $a + p^n \mathbb{Z}_p$ for some $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$ at the remaining p.*

Proof of Claim: First, fix $x_{\text{finite}} \in \mathbb{A}_{\text{finite}}$. It is fixed for the entirety of this proof. Consider all points of the form $x = \{x_\infty, x_{\text{finite}}\}$ which are in the support of h. Each element of this set is contained in a set U with the following properties:

(1) *U is open, and there is a smooth function h_∞^U such that $h(x) = h_\infty^U(x_\infty)$ for all $x \in U$. (From smoothness)*

(2) *$U = U_\infty \cdot \prod_p U_p$ where U is open, $U_p = \mathbb{Z}_p$ for almost all p, and is of the form $a + p^n \mathbb{Z}_p$ for some $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$ at the remaining p. (Every open set contains one of this form.)*

The set of all points of the form $x = \{x_\infty, x_{\text{finite}}\}$ which are in the support of h is a compact set. Using compactness, we get a finite subcover consisting of sets with properties (1) and (2). Let us say that they are $U^{(1)}, \dots U^{(N)}$ and that $U^{(i)} = U_\infty^{(i)} \times \prod_p U_p^{(i)}$. Let $U_p = \cap_{i=1}^N U_p^{(i)}$. Suppose $U_p^{(i)} = a_i + p^{n_i} \mathbb{Z}_p$ for each p, i (So for almost all p, i the value of n_i is 0 and a_i is an integer, so that this is $\mathbb{Z}_p$!) Then $U_p = a + p^n \mathbb{Z}_p$ where $n = \max_i n_i$ and $a = a_{i_0}$ where i_0 is any of the values of i such that $n_i = n$. The sets $U_p^{(i)}$ can not be disjoint since all of the sets

$U^{(i)}_{\text{finite}} := \prod_p U_p^{(i)}$ contain our fixed x_{finite}. If two sets of the form $a + p^n\mathbb{Z}_p$ (different a's same n) intersect, then they are the same).

Now, we simply define $U_{\text{finite}} = \prod_p U_p$, and we also define

$$h_\infty(x_\infty) = h^{U^{(i)}}(x_\infty), \qquad \left(\forall x_\infty \text{ such that } \{x_\infty, x_{\text{finite}}\} \in U^{(i)}\right).$$

One must check that this gives a well defined function, because the sets $U^{(i)}$ overlap. From the definition of $h^{U^{(i)}}$ as in property (1) above, this function may also be described as

$$h_\infty(x_\infty) = h(\{x_\infty, x_{\text{finite}}\}), \qquad \forall\, x_\infty \in \bigcup_{i=1}^{N} U_\infty^{(i)}.$$

From the second description, it is clear that the function is well defined. From the first, it is clear that it is smooth at every point in $U_\infty^{(i)}$. It may be extended to a function on all of $\mathbb{R}$ by setting it equal to zero everywhere else, and one then has

$$h_\infty(x_\infty) = h(\{x_\infty, x_{\text{finite}}\}), \qquad \forall\, x_\infty \in \mathbb{R}.$$

This completes the proof of the claim.

Now we turn to the proof of the main Lemma. The support of h is contained in a set of the form $\prod_v K_v$ where K_v is compact for all v and $K_p = \mathbb{Z}_p$ for almost all $p < \infty$. Let $K_{\text{finite}} = \prod_p K_p \subset \mathbb{A}_{\text{finite}}$. It is a compact set and is covered by the sets U_{finite} from the refined form of smoothness, so there is a finite subcover.

Recall that if $a_1 + p^{n_1}\mathbb{Z}_p$ and $a_2 + p^{n_2}\mathbb{Z}_p$ intersect, then one of them contains the other. If $n_1 = n_2$ they coincide. Otherwise, suppose $n_1 > n_2$. Then the ball $a_2 + p^{n_2}\mathbb{Z}_p$ is also a coset of the ideal $p^{n_2}\mathbb{Z}_p$ and a finite disjoint union of cosets $\alpha + p^{n_1}\mathbb{Z}_p$ with $a_1 + p^{n_1}\mathbb{Z}_p$.

Using these remarks, it is clear that we may subdivide the elements of our finite subcover to obtain a cover with sets which are of the same form as in the claim, *and* pairwise disjoint.

Let us number the sets $U^{(1)}_{\text{finite}}, \ldots, U^{(M)}_{\text{finite}}$, say, and the corresponding functions $h_\infty^{(1)}, \ldots, h_\infty^{(M)}$. Then

$$h(\{x_\infty, x_{\text{finite}}\}) = h_\infty^{(i)}(x_\infty), \qquad \forall\, x_\infty \in \mathbb{R},\ x_{\text{finite}} \in U^{(i)}_{\text{finite}}.$$

But then because the sets are pairwise disjoint this is the same as

$$h(x) = \sum_{i=1}^{M} h_\infty^{(i)}(x_\infty) \cdot 1_{U^{(i)}_{\text{finite}}} = \sum_{i=1}^{M} h_\infty^{(i)}(x_\infty) \cdot \prod_p 1_{U_p^{(i)}}.$$

(Recall that $U_p^{(i)} = \mathbb{Z}_p$ for almost all p for each i.) In this final form, h is seen to be Schwartz, as defined in 1.7.4. $\square$

We shall now present a simple proof of the Fourier expansion (1.8.2) which holds for smooth periodic adelic functions as in proposition 1.8.5. Recall proposition 1.4.5 which states that a fundamental domain for $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ is given by $[0,1) \cdot \prod_p \mathbb{Z}_p$. This allows us to define an integral $\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}}$ as an integral over this fundamental domain.

If f is factorizable, this integral will factor as the infinite product of local integrals $\int_0^1 \cdot \prod_p \int_{\mathbb{Z}_p}$. If D' is any other fundamental domain for $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$, and f is periodic, then it may be shown that $\int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}}$ is also equal to the integral over D'.

Lemma 1.8.9 *Let $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be a smooth adelic function, as in definition 1.8.4, such that the adelic integral*

$$\int_{\mathbb{A}_{\mathbb{Q}}} f(x)dx$$

given in definition 1.7.5 is convergent. Let $D = [0,1)\cdot\prod \mathbb{Z}_p$ denote the fundamental domain for $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ given in 1.4.5. Then

- *for any $\alpha \in \mathbb{Q}$, the function $f \cdot 1_{\alpha+D}$ is integrable,*
- *the infinite sum*

$$\sum_{\alpha\in\mathbb{Q}} \int_{\alpha+D} f(x)\, dx,$$

converges absolutely to

$$\int_{\mathbb{A}_{\mathbb{Q}}} f(x)\, dx,$$

independently of the order in which the sum over $\mathbb{Q}$ is performed.

Proof: The first statement is clear, since $|f \cdot 1_U| \le |f|$ for any U. We have only to observe that for any $\alpha \in \mathbb{Q}$, the set $\alpha + \mathbb{Z}_p$ is compact for all p and equal to $\mathbb{Z}_p$ for almost all p, while the set $\alpha + [0,1)$ is Lebesgue-measurable.

The second statement follows easily from remark (2) after definition 1.7.5. $\square$

Theorem 1.8.10 (Fourier expansion of smooth periodic adelic functions) *Let $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be a smooth periodic adelic function as in definition 1.8.4. Then*

$$\boxed{f(x) = \sum_{\alpha\in\mathbb{Q}} \hat{f}_\alpha \cdot e(\alpha x)}$$

where the above sum converges absolutely for all $x \in \mathbb{A}_{\mathbb{Q}}$ and

$$\boxed{\hat{f}_\alpha \;=\; \int\limits_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} f(x)e(-\alpha x)\, dx \;=\; \hat{h}(\alpha), \qquad (\textit{for all } \alpha \in \mathbb{Q}).}$$

Here h is any adelic Schwartz function such that $f(x) = \sum_{\beta\in\mathbb{Q}} h(x+\beta)$ as in 1.8.5, and $\hat{h}$ is the adelic Fourier transform of h as in definition 1.7.7.

Proof: The proof is presented in 6 steps. By proposition 1.8.4, we may assume that

$$f(x) = \sum_{\beta\in\mathbb{Q}} h(x+\beta) \qquad (x \in \mathbb{A}_{\mathbb{Q}})$$

for some adelic Schwartz function $h : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$. It is enough to give the proof for the case when h is factorizable as in definition 1.7.3.

Step 1: We prove $\hat{f}_\alpha = \hat{h}(\alpha)$.

This follows from the computation shown below. We use the fact that $e(\alpha\beta) = 1$ for $\alpha\beta \in \mathbb{Q}$ (proposition 1.7.2) and the fact that the union of all rational translates of the fundamental domain for $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ is just $\mathbb{A}_\mathbb{Q}$ (proposition 1.4.5).

$$\begin{aligned}\hat{f}_\alpha &= \int\limits_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \sum_{\beta\in\mathbb{Q}} h(x+\beta)e(-\alpha x)\,dx \\ &= \sum_{\beta\in\mathbb{Q}} \int\limits_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} h(x+\beta)e(-\alpha x)\,dx \\ &= \sum_{\beta\in\mathbb{Q}} \int\limits_{-\beta+\ \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} h(x)e(-\alpha x)e(\alpha\beta)\,dx \\ &= \int_{\mathbb{A}_\mathbb{Q}} h(x)e(-\alpha x)\,dx \\ &= \hat{h}(\alpha).\end{aligned}$$

Step 2: We show there exists fixed $N \in \mathbb{Z}$ such that $\hat{f}_\alpha = 0$ unless $\alpha = \frac{n}{N}$ with $n \in \mathbb{Z}$. We may think of the minimal positive N satisfying this condition as the conductor of f.

If p is a prime and h_p is the characteristic function of $\mathbb{Z}_p$, then it follows from proposition 1.6.7 that

$$\int\limits_{\mathbb{Q}_p} h_p(x_p)e_p(-\alpha x_p)\,dx_p = \int\limits_{\mathbb{Z}_p} e_p(-\alpha x_p)\,dx_p = 0$$

unless $|\alpha|_p \le 1$. Since $h_p = 1_{\mathbb{Z}_p}$ is the characteristic function of $\mathbb{Z}_p$ for all but finitely many primes p it follows that $\hat{h}(\alpha) = 0$ unless $\alpha = \frac{n}{N}$ (with $n \in \mathbb{Z}$) where $N = \prod_{i=1}^{\ell} p_i^{a_i}$ and $p_1, p_2, \dots, p_\ell$ are the finitely many primes where $h_{p_i} \ne 1_{\mathbb{Z}_{p_i}}$ for $i = 1, 2, \dots, \ell$. The exponents $a_i \in \mathbb{Z}$ are determined by the fact that each h_{p_i} is a locally constant compactly supported function for $i = 1, 2, \dots, \ell$.

Step 3: Next we show that there exists a fixed constant $C > 0$ (depending at most on f) such that

$$|\hat{f}_{\frac{n}{N}}| < Cn^{-2}$$

where $|\,|$ denotes the ordinary absolute value on $\mathbb{C}$. This will establish the absolute convergence of the Fourier series $\sum_{\alpha\in\mathbb{Q}} \hat{f}_\alpha e(\alpha x)$.

This follows immediately from the fact that the Fourier transform of an adelic Schwartz function is again Schwartz which has rapid decay properties at ∞. From the properties of a Schwartz function, one may actually obtain the stronger bound Cn^{-B} for any fixed constant $B > 0$.

Step 4: It is enough to prove that

$$f(0) \ = \ \sum_{\alpha\in\mathbb{Q}} \hat{f}_\alpha. \tag{1.8.11}$$

To see this fix $x_0 \in \mathbb{A}_\mathbb{Q}$ and define a new function $g(x) = f(x + x_0)$ for $x \in \mathbb{A}_\mathbb{Q}$. Then g is again a periodized Schwartz function, so that by (1.8.11), we have

$$f(x_0) = g(0) = \sum_{\alpha\in\mathbb{Q}} \hat{g}_\alpha.$$

But

$$\hat{g}_\alpha = \int\limits_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} f(x + x_0)e(-\alpha x)\, dx = e(\alpha x_0) \int\limits_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} f(x)e(-\alpha x)\, dx = e(\alpha x_0)\hat{f}_\alpha,$$

from which it follows that

$$f(x_0) = \sum_{\alpha\in\mathbb{Q}} \hat{f}_\alpha e(\alpha x_0).$$

Step 5: It is enough to prove (1.8.11) for functions f which satisfy the condition $f(0) = 0$. If this is not the case, consider the new function $f(x)-f(0)$ which vanishes at 0.

Step 6: We are reduced to proving that

$$\sum_{n\in\mathbb{Z}} \hat{f}_{\frac{n}{N}} = 0 \tag{1.8.12}$$

where f satisfies $f(0) = 0$. Here N is the conductor of f as in Step 2.

Define a new function

$$g(x) = \frac{f(x)}{1 - e\,(x/N)}.$$

By definition, g is again a periodic adelic function. We compute

$$\begin{aligned}
\hat{f}_{\frac{n}{N}} &= \int\limits_{\mathbb{A}_\mathbb{Q}/\mathbb{Q}} f(x)e\left(-\frac{n}{N}x\right)\, dx \\
&= \int\limits_{\mathbb{A}_\mathbb{Q}/\mathbb{Q}} \left(1 - e\left(\frac{x}{N}\right)\right) g(x)e\left(-\frac{n}{N}x\right)\, dx \\
&= \int\limits_{\mathbb{A}_\mathbb{Q}/\mathbb{Q}} g(x)e\left(-\frac{n}{N}x\right)\, dx \ - \int\limits_{\mathbb{A}_\mathbb{Q}/\mathbb{Q}} g(x)e\left(-\frac{n-1}{N}x\right)\, dx \\
&= \hat{g}_{\frac{n}{N}} - \hat{g}_{\frac{n-1}{N}}.
\end{aligned}$$

It follows that

$$\sum_{n\in\mathbb{Z}} \hat{f}_{\frac{n}{N}} = \sum_{n\in\mathbb{Z}} \left(\hat{g}_{\frac{n}{N}} \; - \; \hat{g}_{\frac{n-1}{N}} \right) = 0,$$

since the latter is a telescoping sum where all the terms cancel. □

§1.9 Adelic Poisson summation formula

Let h be an adelic Schwartz function as in definition 1.7.4. The adelic Poisson summation formula states that

(1.9.1)
$$\boxed{\sum_{\alpha\in\mathbb{Q}} h(\alpha) \; = \; \sum_{\alpha\in\mathbb{Q}} \hat{h}(\alpha)}$$

where $\hat{h}$ is the adelic Fourier transform of h as defined in 1.7.7. For applications, we require the following generalization of (1.9.1):

(1.9.2)
$$\boxed{\sum_{\alpha\in\mathbb{Q}} h(\alpha y) \; = \; \frac{1}{|y|_{\mathbb{A}}} \sum_{\alpha\in\mathbb{Q}} \hat{h}\left(\frac{\alpha}{y}\right)}$$

which holds for any idele $y = \{y_\infty, y_2, y_3, \dots\}$ and where $|y|_{\mathbb{A}} = \prod_v |y_v|_v$ is the adelic absolute value.

Proof of (1.9.1): Define the periodized Schwartz function

$$f(x) = \sum_{\alpha\in\mathbb{Q}} h(x+\alpha)$$

as in proposition 1.8.3. The Fourier expansion in theorem 1.8.10 can be applied to f and we obtain

(1.9.3)
$$f(x) = \sum_{\alpha\in\mathbb{Q}} \hat{h}(\alpha) \cdot e(\alpha x).$$

Letting $x = 0$ in (1.9.3) immediately establishes (1.9.1).
□

Proof of (1.9.2): For a fixed idele y, the function $g(x) = h(xy)$ is again an adelic Schwartz function. The result follows on using the relation

$$\hat{g}(x) = \frac{1}{|y|_{\mathbb{A}}} \hat{h}\left(\frac{x}{y}\right).$$

□

Exercises for Chapter 1

1.1 Show that any absolute value on a finite field is trivial.

1.2 Let F be a field and write n for the element of F given by adding 1 to itself n times. Prove that an absolute value $|\cdot|$ is non-archimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}$. **Hint:** *For the sufficiency statement, compare* $|x+y|^n$ *and* $\max(|x|, |y|)^n$.

1.3 For $a \in \mathbb{Q}_p$ and $r > 0$, define $B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$ to be the closed ball of radius r centered at a. For example, $B(a, p^{-m}) = a + p^m\mathbb{Z}_p$.

- **(a)** For any $b \in B(a, r)$, show that $B(a, r) = B(b, r)$. That is, every point of a closed ball in $\mathbb{Q}_p$ can act as the center.
- **(b)** Show that any two closed balls in $\mathbb{Q}_p$ are either disjoint, or else one contains the other.
- **(c)** Show that closed balls are also open in the p-adic topology.
- **(d)** Find a set that is closed but not open, and one that is open but not closed.

1.4 Is the rational number $\frac{1}{1-p}$ an element of $\mathbb{Z}_p$? What is its p-adic power series expansion?

1.5 Does $\sqrt{-1}$ exist in $\mathbb{Q}_3$? In $\mathbb{Q}_5$? In $\mathbb{Q}_2$? **Hint:** *Said another way, can one solve the equation* $x^2 = -1$ *in these fields?*

1.6 This exercise characterizes all locally constant and compactly supported functions on $\mathbb{Q}_v$ for $v \leq \infty$.

- **(a)** For p a prime, show that any compact open subset of $\mathbb{Q}_p$ is just a finite union of neighborhoods of the form $a + p^m\mathbb{Z}_p$, where $a \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$.
- **(b)** Suppose $h : \mathbb{Q}_p \to \mathbb{C}$ is locally constant and compactly supported. Prove that h is a finite linear combination of characteristic functions of the form $1_{a+p^m\mathbb{Z}_p}$.
- **(c)** If $h : \mathbb{R} \to \mathbb{C}$ is locally constant and compactly supported, prove that h is identically zero.

1.7 This exercise proves that the topology on $\mathbb{A}_{\mathbb{Q}}^{\times}$ is strictly finer than the subspace topology induced by $\mathbb{A}_{\mathbb{Q}}$.

- **(a)** Show that if $V \subset \mathbb{A}_{\mathbb{Q}}$ is an open subset, then $V \cap \mathbb{A}_{\mathbb{Q}}^{\times}$ is open in the topology of $\mathbb{A}_{\mathbb{Q}}^{\times}$.
- **(b)** Show that the sets in the basis for the topology specified after 1.3.2 are not open in the subspace topology of $\mathbb{A}_{\mathbb{Q}}^{\times} \subset \mathbb{A}_{\mathbb{Q}}$.
- **(c)** Define an injective map $i\colon \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$ via the rule $i(x) = (x, x^{-1})$. Show that the topology on $\mathbb{A}_{\mathbb{Q}}^{\times}$ coincides with the subspace topology of $i(\mathbb{A}_{\mathbb{Q}}^{\times}) \subset \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$.

1.8 Consider the adelic function $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ defined by $f = \prod_v f_v$, where

$$f_v(x_v) = \begin{cases} \exp(-\pi x_\infty^2) & \text{if } v = \infty \\ 1_{\mathbb{Z}_p}(x_p) + 1_{p^{-1}+p^2\mathbb{Z}_p}(x_p) & \text{if } v = p \text{ is prime} \end{cases}.$$

Show that the Fourier inversion $\hat{f}$ is a well-defined function, but that $\hat{f}$ is not Schwartz. Why does this not contradict Theorem 1.7.6? **Hint:** *It may be useful to know that an infinite product $\prod(1 + a_n)$ converges if the series $\sum a_n$ converges absolutely.*

1.9 Show that a smooth adelic function $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ is continuous (for the adelic topology).

1.10 The goal of this exercise is to illustrate a technical detail from step 6 of the proof of Theorem 1.8.9. Suppose that $h(x) = \prod_v h_v(x)$ is a factorizable adelic Schwartz function, and suppose $f(x) = \sum_{\alpha\in\mathbb{Q}} h(x+\alpha)$ has conductor N (as in step 2 of the proof of Theorem 1.8.9). We will show there is a smooth periodic adelic function $g : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ such that

$$(*) \qquad g(x) = \frac{f(x) - f(0)}{1 - e(x/N)},$$

whenever this expression makes sense. As each h_v is a finite linear combination of characteristic functions, one can observe that it suffices in the proof of Theorem 1.8.9 to assume that each finite factor of h is of the form $h_p = 1_{a+p^m\mathbb{Z}_p}$ for some $a \in \mathbb{Q}_p$ and some integer m (that depend on the prime p).

- **(a)** Prove that the function $f(Nx)$ is smooth and periodic with conductor 1. Deduce that it suffices to prove $(*)$ when $N = 1$.
- **(b)** Deduce from part (a) that it suffices to prove $(*)$ when $N = 1$ and

  $$h(x) = h_\infty(x_\infty) \prod_p 1_{a(p)+\mathbb{Z}_p}(x_p),$$

  for some $a(p) \in \mathbb{Q}_p$ such that $a(p) = 0$ for all but finitely many p. By replacing $h(x)$ with $h(x + \beta)$ for a clever choice of rational number β, show that we may even assume $a(p) = 0$ for all primes p.
- **(c*)** Assume now that $h = h_\infty \prod_p 1_{\mathbb{Z}_p}$ and $N = 1$. When $e(x) \neq 1$, define $g(x)$ by the formula $(*)$; when $e(x) = 1$, set

  $$g(x) = \frac{-1}{2\pi i} \sum_{n\in\mathbb{Z}} h'_\infty(n).$$

  Show that $g(x)$ is smooth and periodic.

1.11* In this exercise, we develop a few of the interesting analogies between the rational numbers $\mathbb{Q}$ and the field of rational functions $\mathbb{F}_p(T)$. It will require a little more in the way of abstract algebra than some of the other exercises. Let p be a prime number, and $\mathbb{F}_p$ the field of p elements. The **rational function field** with coefficients in $\mathbb{F}_p$ is the field of rational functions $\mathbb{F}_p(T)$. An element $f \in \mathbb{F}_p(T)$ is of the form

$$f(T) = \frac{P_1(T)}{P_2(T)}$$

for some polynomials P_1, P_2 with coefficients in $\mathbb{F}_p$, where P_2 is not the zero polynomial. If you haven't seen this concept before, verify that $\mathbb{F}_p(T)$ is a field.

(a) Prove that any nontrivial absolute value on $\mathbb{F}_p(T)$ is equivalent (in the sense of definition 1.1.3) to one of $|\cdot|_Q$ or $|\cdot|_\infty$, defined as follows:

(i) Fix a monic irreducible polynomial $Q \in \mathbb{F}_p[T]$. Since $\mathbb{F}_p[T]$ is a unique factorization domain, any nonzero rational function $f \in \mathbb{F}_p(T)$ can be written uniquely as $f = Q^r g$ for some integer r and some rational function g such that Q does not divide the numerator or denominator of g. Define an absolute value on $\mathbb{F}_p(T)$ by $|f|_Q = p^{-r\deg(Q)}$ and $|0|_Q = 0$.

(ii) Let $f = P_1/P_2$ be a nonzero rational function. Define

$$|f|_\infty = p^{\deg(P_1)-\deg(P_2)},$$

where deg denotes the degree of a polynomial. Set $|0|_\infty = 0$.

(b) Let Q be a monic irreducible polynomial with coefficients in $\mathbb{F}_p$, let $d = \deg(Q)$, and let $q = p^d$. (Here we will always assume that such a Q is nonconstant.) Define $\mathbb{F}_q((Q))$ to be the **field of formal Laurent series** in the variable Q with coefficients in $\mathbb{F}_q$:

$$\mathbb{F}_q((Q)) = \left\{ \sum_{i=N}^{\infty} a_i Q^i \;\middle|\; N \in \mathbb{Z},\ a_i \in \mathbb{F}_q \right\}.$$

This field has an absolute value defined by

$$\left| \sum_{i=N}^{\infty} a_i Q^i \right|_Q = q^{-N} \qquad (a_N \neq 0).$$

Prove that the completion of $\mathbb{F}_p(T)$ with respect to the absolute value $|\cdot|_Q$ can be identified with $\mathbb{F}_q((Q))$. **Hint:** *The field* $\mathbb{F}_q$ *arises naturally in this context as* $\mathbb{F}_q = \mathbb{F}_p[T]/(Q(T))$.

(c) Let $\mathbb{F}_q((1/T))$ be the ring of formal Laurent series in the variable $1/T$. This field has an absolute value defined by

$$\left| \sum_{i=N}^{\infty} a_i \left(\frac{1}{T}\right)^i \right|_\infty = p^{-N} \qquad (a_N \neq 0).$$

Prove that the completion of $\mathbb{F}_p(T)$ with respect to the absolute value $|\cdot|_\infty$ can be identified with $\mathbb{F}_p((1/T))$.

(d) Let the symbol v denote either ∞ or a monic irreducible polynomial $Q \in \mathbb{F}_p[T]$. Show that the following product formula holds for any nonzero rational function $f \in \mathbb{F}_p(T)$:

$$\prod_v |f|_v = 1.$$

(e) For each monic irreducible polynomial $Q \in \mathbb{F}_p[T]$, we define a subring of the completion by

$$\mathcal{O}_Q = \Big\{ f \in \mathbb{F}_q((Q)) \;\Big|\; |f|_Q \leq 1 \Big\}.$$

Similarly, we have

$$\mathcal{O}_\infty = \Big\{ f \in \mathbb{F}_p((1/T)) \;\Big|\; |f|_\infty \leq 1 \Big\}.$$

Define the rational function field adeles to be the restricted product (relative to the subgroups $\mathcal{O}_Q$) of

$$\mathbb{A}_{\mathbb{F}_p(T)} = \mathbb{F}_p((1/T)) \times \prod_{\substack{Q \in \mathbb{F}_p[T] \\ \text{monic} \\ \text{irreducible}}} \mathbb{F}_q((Q)).$$

Show that a fundamental domain for the additive action of $\mathbb{F}_p(T)$ on $\mathbb{A}_{\mathbb{F}_p(T)}$ is given by

$$\left(\frac{1}{T} \mathcal{O}_\infty \right) \times \prod_{\substack{Q \in \mathbb{F}_p[T] \\ \text{monic} \\ \text{irreducible}}} \mathcal{O}_Q.$$

1.12* In this exercise we indicate all of the necessary adjustments in order to make sense of Fourier transforms, the Fourier inversion theorem, and the Poisson summation formula for the function field adeles $\mathbb{A}_{\mathbb{F}_p(T)}$. Parts **(d-g)** are essentially verifying that the proofs in the case of $\mathbb{A}_\mathbb{Q}$ carry over to the function field setting. We continue with the notation from Exercise 1.11.

(a) Write $x = \sum_{i=N}^{\infty} a_i (1/T)^i$ for an element of $\mathbb{F}_p((1/T))$. Define a function $\psi_\infty : \mathbb{F}_p((1/T)) \to \mathbb{C}^\times$ by

$$\psi_\infty(x) = e^{-2\pi i a_1/p}.$$

The map ψ_∞ is well-defined if we interpret a_1 as an integer modulo p. Show that ψ_∞ is a unitary character on the additive group of $\mathbb{F}_p((1/T))$ and that $\psi_\infty\big((1/T)^2 \mathcal{O}_\infty\big) = 1$.

(b) Let Q be a monic irreducible polynomial in $\mathbb{F}_p[T]$. Write $x = \sum_{i=N}^{\infty} a_i Q^i$ for an element of $\mathbb{F}_q((Q))$. Via the identification $\mathbb{F}_q = \mathbb{F}_p[T]/(Q(T))$, we can express the coefficient $a_{-1} \in \mathbb{F}_q$ as

$$a_{-1} \equiv c_0 + c_1 T + \cdots + c_{\deg(Q)-1} T^{\deg(Q)-1} \pmod{Q(T)},$$

where each $c_j \in \mathbb{F}_p$. Define a function $\psi_Q : \mathbb{F}_q((Q)) \to \mathbb{C}^\times$ by

$$\psi_Q(x) = e^{2\pi i c_0/p}.$$

Show that ψ_Q is a unitary character on the additive group of $\mathbb{F}_q((Q))$ and that $\psi_Q(\mathcal{O}_Q) = 1$.

(c) Now write $x = (x_v) \in \mathbb{A}_{\mathbb{F}_p(T)}$. Define an adelic unitary character $\psi : \mathbb{A}_{\mathbb{F}_p(T)} \to \mathbb{C}^\times$ by the formula

$$\psi(x) = \prod_v \psi_v(x_v),$$

where the local characters ψ_v were defined in parts **(a)** and **(b)** above. Show that ψ is nontrivial and that $\psi(f) = 1$ for every $f \in \mathbb{F}_p(T)$. **Hint:** *Follow the strategy of Theorem 1.7.2.*

(d) Normalize the Haar measure on $\mathbb{F}_p((1/T))$ so that $\mu_{\text{Haar}}(\mathcal{O}_\infty) = p$, and let $dx = d\mu_{\text{Haar}}(x)$. A Schwartz function on $\mathbb{F}_p((1/T))$ is a locally constant compactly supported function. Define the Fourier transform of a Schwartz function $f : \mathbb{F}_p((1/T)) \to \mathbb{C}$ by

$$\hat{f}(x) = \int_{\mathbb{F}_p((1/T))} f(y)\psi_\infty(-xy)dy.$$

Show that $\hat{f}$ is Schwartz and that the Fourier inversion formula holds:

$$\hat{\hat{f}}(x) = f(-x) \qquad (x \in \mathbb{F}_p((1/T))).$$

(e) Again, let Q be a monic irreducible polynomial in $\mathbb{F}_p[T]$. Normalize the Haar measure on $\mathbb{F}_q((Q))$ so that $\mu_{\text{Haar}}(\mathcal{O}_Q) = 1$, and let $dx = d\mu_{\text{Haar}}(x)$. Define the Fourier transform of a Schwartz function $f : \mathbb{F}_q((Q)) \to \mathbb{C}$ by

$$\hat{f}(x) = \int_{\mathbb{F}_q((Q))} f(y)\psi_Q(-xy)dy.$$

Show that $\hat{f}$ is Schwartz and that the Fourier inversion formula holds as in part **(d)**.

(f) Conclude that we may define a Fourier transform on adelic Schwartz functions (as in Definition 1.7.4) and that Fourier inversion holds as in Theorem 1.7.6.

(g) Define a smooth adelic function on $\mathbb{A}_{\mathbb{F}_p(T)}$ to be the same thing as an adelic Schwartz function. Modify the statements and proofs in §1.8 and §1.9 to conclude that the Poisson sumation formula holds for function field adeles. **Hint:** *Replace* $\mathbb{Q}$ *by* $\mathbb{F}_p(T)$ *and replace* $\mathbb{Z}$ *by* $\mathbb{F}_p[T]$*. This is especially important in step 2 of the proof of Theorem 1.8.9.*

1.13* The goal of this exercise is to sketch the proof of a special case of the statement "The Poisson summation formula for function field adeles yields the Riemann-Roch theorem for curves." We continue with the notation from Exercises 1.11 and 1.12. Recall that the symbol v is allowed to denote either ∞ or a nontrivial monic irreducible polynomial $Q \in \mathbb{F}_p[T]$. The **degree** of v, denoted $\deg(v)$, is given by

$$\deg(v) = \begin{cases} 1 & \text{if } v = \infty \\ \deg(Q) & \text{if } v = Q \text{ is a monic irreducible polynomial.} \end{cases}$$

Define a **divisor** D to be a finite formal linear combination of all possible symbols v with coefficients in the integers:

$$D = \sum_v n_v.v \qquad (n_v \in \mathbb{Z}).$$

The **degree** of the divisor D is the integer $\deg(D) = \sum_v n_v \deg(v)$. We let

$$L(D) = \left\{ f \in \mathbb{F}_p(T) \;\middle|\; |f|_v \le p^{n_v \deg(v)} \text{ for all } v \right\}.$$

A rational function $f \in L(D)$ is bounded v-adically in terms of the divisor D. We will see that $L(D)$ is an $\mathbb{F}_p$-vector space, and the essence of the Riemann-Roch theorem is that we can calculate its dimension.

(a) Verify that $L(D)$ is a finite-dimensional $\mathbb{F}_p$-vector space.

(b) Write $x = (x_v)$ for an element of $\mathbb{A}_{\mathbb{F}_p(T)}$. Define an adelic Schwartz function $h_D : \mathbb{A}_{\mathbb{F}_p(T)} \to \mathbb{C}$ by the rule

$$h_D(x) = \begin{cases} 1 & \text{if } |x_v|_v \le p^{n_v \deg(v)} \text{ for all } v \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that

$$\sum_{f \in \mathbb{F}_p(T)} h_D(f) = p^{\dim L(D)}.$$

(c) Show that the Fourier transform $\hat{h}_D$ is given by

$$\hat{h}_D(x) = \begin{cases} p^{\deg(D)+1} & \text{if } |x_\infty|_\infty \le p^{-n_\infty - 2} \\ & \qquad \text{and } |x_v|_v \le p^{-n_v \deg(v)} \text{ for all } v \ne \infty \\ 0 & \text{otherwise.} \end{cases}$$

Hint: *As h_D is factorizable, one can compute the Fourier transforms for each v separately. Now follow Proposition 1.6.7.*

(d) Define a divisor by $K = -2.\infty$. Show that

$$\sum_{f \in \mathbb{F}_p(T)} \hat{h}_D(f) = p^{\dim L(K-D) + \deg(D) + 1}.$$

(e) By the previous exercise, we know that the Poisson summation formula holds in this context. Deduce the Riemann-Roch formula for the function field $\mathbb{F}_p(T)$:

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1.$$