CHAPTER II

AUTOMORPHIC REPRESENTATIONS AND L-FUNCTIONS FOR $\mathbf{GL}(\mathbf{1}, \mathbb{A}_\mathbb{Q})$

§2.1 Automorphic forms for $\mathbf{GL}(\mathbf{1}, \mathbb{A}_\mathbb{Q})$

Let $\mathbb{A}_\mathbb{Q}$ denote the adele ring over $\mathbb{Q}$ as in definition 1.3.1. The key point is that $GL(1, \mathbb{A}_\mathbb{Q})$ is just $\mathbb{A}_\mathbb{Q}^\times$, the multiplicative subgroup of ideles of $\mathbb{Q}$. In conformity with modern notation we shall use the notation

$$g = \{g_\infty, g_2, g_3, \dots\}$$

to denote an element of the group $GL(1, \mathbb{A}_\mathbb{Q})$. Here $g_v \in \mathbb{Q}_v^\times$ for all v and $g_p \in \mathbb{Z}_p^\times$ for all but finitely many finite primes p.

The multiplicative group $\mathbb{Q}^\times$ is diagonally embedded in $\mathbb{A}_\mathbb{Q}^\times$ and acts on $\mathbb{A}_\mathbb{Q}^\times$ by left multiplication. Proposition 1.4.6 tells us that a fundamental domain for this action is given by

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \; = \; (0, \infty) \cdot \prod_p \mathbb{Z}_p^\times,$$

where the product is over all finite primes p.

An idelic function $f : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}$ is said to be factorizable if it is determined by local functions $f_v : \mathbb{Q}_v^\times \to \mathbb{C}$ $(\forall v \le \infty)$, where $f_p \equiv 1$ on $\mathbb{Z}_p^\times$ for all but finitely many finite primes p, and where

$$f(g) = \prod_{v \le \infty} f_v(g_v), \; \forall g = \{g_\infty, g_2, g_3, \dots\} \in \mathbb{A}_\mathbb{Q}^\times. \tag{2.1.1}$$

Note that this definition is slightly different than the notion of "factorizability of adelic functions" which was given in 1.7.3.

Definition 2.1.2 (Unitary Hecke character of $\mathbb{A}_\mathbb{Q}^\times$) *A Hecke character of $\mathbb{A}_\mathbb{Q}^\times$ is defined to be a continuous homomorphism*

$$\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times.$$

A Hecke character is said to be unitary if all its values have absolute value 1. A unitary Hecke character of $\mathbb{A}_\mathbb{Q}^\times$ is characterized by the following four properties:

- **(i)** $\omega(gg') = \omega(g)\omega(g'), \quad \forall g, g' \in \mathbb{A}_\mathbb{Q}^\times$;
- **(ii)** $\omega(\gamma g) = \omega(g), \quad \forall \gamma \in \mathbb{Q}^\times, \; \forall g \in \mathbb{A}_\mathbb{Q}^\times$;
- **(iii)** ω *is continuous at* $\{1, 1, 1, \dots\}$.
- **(iv)** $|\omega| = 1$.

Definition 2.1.3 (Moderate growth) *We say that a function $\phi : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}$ is of moderate growth if, for each $g = \{g_\infty, g_2, g_3, \dots\} \in \mathbb{A}_\mathbb{Q}^\times$, there exist positive constants C and M such that*

$$\phi\Big(\{tg_\infty, g_2, g_3, \dots\}\Big) < C\,(1 + |t|_\infty)^M$$

for all $t \in \mathbb{R}$.

Definition 2.1.4 (Automorphic form) *Fix a unitary Hecke character ω as in (2.1.2). An automorphic form for $GL(1, \mathbb{A}_\mathbb{Q})$ with character ω is a function*

$$\phi : GL(1, \mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

which satisfies the following conditions:

(1) $\phi(\gamma g) = \phi(g), \quad \forall g \in \mathbb{A}_\mathbb{Q}^\times, \ \forall \gamma \in \mathbb{Q}^\times;$

(2) $\phi(zg) = \omega(z)\phi(g), \quad \forall g \in \mathbb{A}_\mathbb{Q}^\times, \ \forall z \in \mathbb{A}_\mathbb{Q}^\times;$

(3) *ϕ is of moderate growth as in definition 2.1.3.*

Let $\mathcal{S}_\omega$ denote the set of all automorphic forms for $GL(1, \mathbb{A}_\mathbb{Q})$ with character ω as in definition 2.1.4. If $c_1, c_2 \in \mathbb{C}$, are arbitrary complex constants, and $\phi_1, \phi_2 \in \mathcal{S}_\omega$, then it is easy to see that $c_1\phi_1 + c_2\phi_2$ is again automorphic with character ω. The space $\mathcal{S}_\omega$ is, therefore, a vector space over $\mathbb{C}$. Setting $g = \{1, 1, 1, \ \dots\}$ it immediately follows from definition 2.1.4 (2) that $\phi(z) = c\omega(z)$, with $c = \phi(\{1, 1, 1, \dots\})$. Thus $\mathcal{S}_\omega$ is a one-dimensional space. The reader may ask why we simply do not define an automorphic form as a Hecke character as in (2.1.2)? The reason is that we want to give a uniform definition of automorphic form for $GL(n, \mathbb{A}_\mathbb{Q})$ that holds for all $n = 1, 2, 3, \dots$ In the case of $n = 1$, definition 2.1.4 (2) becomes superfluous since z, g both lie in the same space. This is not the case for $n > 1$ as we shall see later.

Definition 2.1.4 may seem rather imposing at first sight but it turns out that the automorphic forms for $GL(1, \mathbb{A}_\mathbb{Q})$ are just classical Dirichlet characters in disguise. For a fixed integer $q > 1$, a Dirichlet character $\chi \pmod q$ is a homomorphism

$$\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times. \tag{2.1.5}$$

That is,

$$\chi(ab) = \chi(a)\chi(b), \quad \forall a, b \in (\mathbb{Z}/q\mathbb{Z})^\times. \tag{2.1.6}$$

Because $a^{\varphi(q)} = 1$ for all $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, such a function must take values in the $\varphi(q)$th roots of unity. In particular,

$$|\chi(a)| = 1, \quad \forall a \in (\mathbb{Z}/q\mathbb{Z})^\times.$$

Here $\varphi(q)$ is Euler's φ function.

It is standard practice in analytic number theory (see [Davenport, 2000]) to lift a Dirichlet character χ to $\mathbb{Z}$ by defining a new function $\chi_1 : \mathbb{Z} \to \mathbb{C}$ which satisfies

$$\begin{aligned}
&\chi_1(a) = 0, \ \ \forall a \in \mathbb{Z} \ \text{ with } (a,q) \neq 1;\\
&\chi_1(a+mq) = \chi(a), \ \ \forall a, m \in \mathbb{Z} \ \text{ with } (a,q) = 1,\\
&\chi_1(ab) = \chi_1(a)\chi_1(b), \ \ \forall a, b \in \mathbb{Z}.
\end{aligned}$$

We shall follow the standard practice of denoting the character χ_1 by the symbol χ. Remarkably, it is also possible to lift χ to the idele group $\mathbb{A}_{\mathbb{Q}}^\times$. Since $\mathbb{A}_{\mathbb{Q}}^\times$ is a purely multiplicative group, the value of the lifted character can never be 0. The result is an automorphic form as in definition 2.1.4. We now explicitly describe and prove the existence of this lifting which was found by Tate and appeared in his thesis [Tate, 1950].

Definition 2.1.7 (Idelic lift of a Dirichlet character) *Let χ be a Dirichlet character of conductor p^f as in (2.1.5), (2.1.6) where p^f is a fixed prime power. We define the idelic lift of χ to be the unitary Hecke character $\chi_{\text{idelic}} : \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ defined as*

$$\chi_{\text{idelic}}(g) = \chi_\infty(g_\infty)\cdot\chi_2(g_2)\cdot\chi_3(g_3)\cdots, \qquad \big(g = \{g_\infty, g_2, g_3, \dots\} \in \mathbb{A}_{\mathbb{Q}}^\times\big),$$

where

$$\chi_\infty(g_\infty) = \begin{cases} 1, & \chi(-1) = 1,\\ 1, & \chi(-1) = -1,\ g_\infty > 0,\\ -1, & \chi(-1) = -1,\ g_\infty < 0,\end{cases}$$

and where

$$\chi_v(g_v) = \begin{cases} \chi(v)^m, & \textit{if } g_v \in v^m\mathbb{Z}_v^\times \textit{ and } v \neq p,\\ \chi(j)^{-1}, & \textit{if } g_v \in p^k\left(j + p^f\mathbb{Z}_p\right) \textit{ with } j,k \in \mathbb{Z},\ (j,p) = 1 \textit{ and } v = p.\end{cases}$$

To see that definition 2.1.7 actually defines a unitary Hecke character satisfying (2.1.2) we make the following observations. First of all, for every prime $v \le \infty$, it is clear from the definition that

$$\chi_v(g_v g_v') = \chi_v(g_v)\cdot\chi_v(g_v')$$

for all g_v, $g_v' \in \mathbb{Q}_v^\times$. Consequently, χ_{idelic} must satisfy (2.1.2) (i). Secondly, if $\ell \neq p$ is any finite prime, then $\chi_\ell(\ell) = \chi(\ell)$. Also $\chi_p(\ell) = \chi(\ell)^{-1}$, $\chi_v(p) = 1$ for any finite prime v and $\chi_v(\ell) = 1$ for any finite prime $v \neq \ell$ and $v \neq p$. It follows that

$$\chi_{\text{idelic}}(\ell) = 1$$

for all primes ℓ. Also $\chi_{\text{idelic}}\big(\{-1,-1,-1,\ \dots,\ \}\big) = \chi_{\text{idelic}}\big(\{1,1,1,\ \dots,\ \}\big) = 1.$

When combined with (2.1.2) (i), this establishes (2.1.2) (ii). Thirdly, we can see directly that the kernel of χ_{idelic} is an open neighborhood of $\{1,1,1,\dots\}$. It is also obvious that $|\chi_{\text{idelic}}| = 1$ since χ has this property on $(\mathbb{Z}/p^n\mathbb{Z})^\times$. The above observations establish that χ_{idelic} is indeed a Hecke character satisfying the four conditions of (2.1.2).

More generally, every Dirichlet character χ of conductor $q = \prod_{i=1}^{r} p_i^{f_i}$, where $p_1, p_2, \dots p_r$ are distinct primes and $f_1, f_2, \dots f_r \geq 1$ can be factored as

$$\chi = \prod_{i=1}^{r} \chi^{(i)},$$

where $\chi^{(i)}$ is a Dirichlet character of conductor $p_i^{f_i}$. It follows that χ may be lifted to a Hecke character χ_{idelic} on $\mathbb{A}_{\mathbb{Q}}^{\times}$ where

$$\chi_{\text{idelic}} = \prod_{i=1}^{r} \chi^{(i)}_{\text{idelic}}. \tag{2.1.8}$$

Theorem 2.1.9 *Every automorphic form ϕ on $GL(1, \mathbb{A}_{\mathbb{Q}})$, as in definition 2.1.4, can be uniquely expressed in the form*

$$\phi(g) = c \cdot \chi_{\text{idelic}}(g) \cdot |g|_{\mathbb{A}}^{it}, \qquad \left(\forall g \in \mathbb{A}_{\mathbb{Q}}^{\times}\right),$$

where $c \in \mathbb{C}$, $t \in \mathbb{R}$, are fixed constants, and χ_{idelic} is an idelic lift of a fixed Dirichlet character χ as in definition 2.1.7 and (2.1.8). Here, if $g = \{g_\infty, g_2, g_3, \dots\}$, then $|g|_{\mathbb{A}} = \prod\limits_{v \leq \infty} |g_v|_v$ is the idelic absolute value.

Proof: It follows from definition 2.1.4 that we may take

$$\phi(g) = c \cdot \omega(g), \qquad (\forall g \in \mathbb{A}_{\mathbb{Q}}^{\times})$$

with $c = \phi\big(\{1, 1, 1, \dots\}\big)$, and where ω is a unitary Hecke character satisfying (2.1.2). For each prime $v \leq \infty$, consider the embedding

$$i_v(g_v) = \{1, \dots, 1, \underbrace{g_v}_{v^{th} \text{ position}}, 1 \dots, 1\}, \qquad g_v \in \mathbb{Q}_v.$$

Then, if we define

$$\omega_v(g_v) := \omega(i_v(g_v)), \qquad (\forall g_v \in \mathbb{Q}_v)$$

then ω_v is a character of $\mathbb{Q}_v^{\times}$ for every prime $v \leq \infty$.

Furthermore,

$$\omega(g) = \prod_{v \leq \infty} \omega_v(g_v) \tag{2.1.10}$$

where $\omega_p(g_p) \equiv 1$, $(\forall g_p \in \mathbb{Z}_p^{\times})$ for all but finitely many finite primes p. The Hecke characters ω can then be determined if we can classify the local characters ω_v for all primes v.

First of all, every unitary continuous multiplicative character $\omega_\infty : \mathbb{R}^{\times} \to \mathbb{C}^{\times}$ is of the form

$$\omega_\infty(g_\infty) = |g_\infty|_\infty^{it}, \qquad \text{or} \qquad \omega_\infty(g_\infty) = |g_\infty|_\infty^{it} \cdot \text{sign}(g_\infty), \tag{2.1.11}$$

for some fixed constant $t \in \mathbb{R}$, where $\text{sign}(g_\infty) = \begin{cases} +1, & \text{if } g_\infty > 0, \\ -1, & \text{if } g_\infty < 0. \end{cases}$

To continue the proof and classify the characters ω_v for $v < \infty$, we introduce some definitions.

Definition 2.1.12 (Unramified local character) *Fix a finite prime p. A local character ω_p, occurring in the decomposition (2.1.10), is said to be unramified if $\omega_p(u) = 1$ for all $u \in \mathbb{Z}_p^\times$. At ∞, the local character $|g_\infty|_\infty^{it}$ (for some fixed $t \in \mathbb{R}$) is said to be unramified.*

Fix a finite prime p. The unramified local characters of $\mathbb{Q}_p^\times$ are easy to describe. Every $g_p \in \mathbb{Q}_p^\times$ is an element of $p^m\mathbb{Z}_p^\times$ for some integer m. Let $g_p = p^m \cdot u$ with $u \in \mathbb{Z}_p^\times$. Then if ω_p is unramified, it follows that

$$\omega_p(g_p) = \omega_p(p^m \cdot u) = \omega_p(p^m) = \omega_p(p)^m. \tag{2.1.13}$$

So once we know the value of ω_p at the one point p, we know its value everywhere.

Definition 2.1.14 (Ramified local character and its conductor) *Fix a finite prime p. We say a local character ω_p, occurring in the decomposition (2.1.10), is ramified if $\omega_p(u) \neq 1$ for some $u \in \mathbb{Z}_p^\times$. The conductor of ω_p is defined to be p^k where k is the smallest positive integer such that $1 + p^k\mathbb{Z}_p$ is contained in the kernel of ω_p. We say the local character ω_∞ is ramified if $\omega_\infty(u) = -\omega_\infty(-u)$ for all $u \in \mathbb{R}^\times$.*

Fix a finite prime p. The ramified local characters of $\mathbb{Q}_p^\times$ can be described as follows. Let $g_p = p^m \cdot u$ with $u \in \mathbb{Z}_p^\times$. Then

$$\omega_p(g_p) = \omega_p(p^m \cdot u) = \omega_p(p)^m \omega_p(u).$$

Since $\omega_p(u) \neq 1$ for some $u \in \mathbb{Z}_p^\times$ it follows, by continuity, that the kernel of ω_p contains an open subgroup of the form $1 + p^k\mathbb{Z}_p$. We shall assume k is minimal and term p^k the conductor of ω_p.

Claim: $\mathbb{Z}_p^\times / \left(1 + p^k\mathbb{Z}_p\right) \cong \left(\mathbb{Z}/p^k\mathbb{Z}\right)^\times$. *To see this note that the multiplicative cosets in $\mathbb{Z}_p^\times / \left(1 + p^k\mathbb{Z}_p\right)$ are all of the form*

$$j\left(1 + p^k\mathbb{Z}_p\right) = j + p^k\mathbb{Z}_p, \quad (1 \le j < p^k,\ (j, p) = 1).$$

Furthermore

$$j\left(1 + p^k\mathbb{Z}_p\right) \cdot j'\left(1 + p^k\mathbb{Z}_p\right) = (j \cdot j')\left(1 + p^k\mathbb{Z}_p\right)$$

from which it follows that we may assume $j, j', (j \cdot j') \in \left(\mathbb{Z}/p^k\mathbb{Z}\right)^\times$.

The above claim implies that the ramified local character ω_p must satisfy:

$$\omega_p(g_p) = \omega_p(p)^m \chi^{(p^k)}(j)^{-1}, \tag{2.1.15}$$

for some fixed Dirichlet character $\chi^{(p^k)}$ of $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^\times$, for all $j \in \left(\mathbb{Z}/p^k\mathbb{Z}\right)^\times$, and whenever $g_p \in p^m \cdot \left(j + p^k\mathbb{Z}_p\right)$.

The relations (2.1.10), (2.1.11), (2.1.13) and (2.1.15) specify all the possible local characters that may occur in the factorization (2.1.10) of ω. Now the local characters ω_v must be chosen so that

$$\omega(\gamma g) = \prod_{v \le \infty} \omega_v(\gamma g_v) = \omega(g) \tag{2.1.16}$$

for all $g \in \mathbb{A}_{\mathbb{Q}}^{\times}$, and for all $\gamma \in \mathbb{Q}^{\times}$. The relation $\omega(\gamma g) = \omega(\gamma)\omega(g) = \omega(g)$ implies that $\omega(\gamma) = 1$ for all $\gamma \in \mathbb{Q}^{\times}$.

Let $S = \{p_1, p_2, \ldots, p_r\}$ denote the finite set of primes where ω_{p_i} is ramified with conductor $p_i^{k_i}$ (for $i = 1, 2, \ldots, r$). Define

$$\chi = \prod_{i=1}^{r} \chi^{(p_i^{k_i})}$$

where each $\chi^{(p_i^{k_i})}$ is determined by (2.1.15) for $i = 1, 2, \ldots, r$. Let $\ell > 0$ be a prime where $\ell \notin S$. Then combining (2.1.16) with (2.1.11), (2.1.14), (2.1.15), it follows that

$$\omega(\ell) \;=\; 1 \;=\; \ell^{it} \cdot \omega_\ell(\ell) \cdot \prod_{i=1}^{r} \chi^{(p_i^{k_i})}(\ell)^{-1}.$$

This forces

$$\omega_\ell(\ell) = \ell^{-it} \prod_{i=1}^{r} \chi^{(p_i^{k_i})}(\ell) = \ell^{-it} \cdot \chi(\ell). \tag{2.1.17}$$

On the other hand, if $\ell \in S$, we have

$$\omega(\ell) \;=\; 1 \;=\; \ell^{it} \cdot \omega_\ell(\ell) \cdot \prod_{\substack{i=1 \\ p_i \ne \ell}}^{r} \chi^{(p_i^{k_i})}(\ell)^{-1}.$$

So in this case,

$$\omega_\ell(\ell) = \ell^{-it} \prod_{\substack{i=1 \\ p_i \ne \ell}}^{r} \chi^{(p_i^{k_i})}(\ell)^{-1}. \tag{2.1.18}$$

Finally we consider the case where $\gamma = -1$ in (2.1.16). It follows that

$$\omega(-1) = 1 = \omega_\infty(-1) \prod_{i=1}^{r} \chi^{(p^{k_i})}(-1)^{-1} = \omega_\infty(-1)\chi(-1). \tag{2.1.19}$$

Equations (2.1.17), (2.1.18), (2.1.19) together with definition 2.1.7 imply theorem 2.1.9.

□

§2.2 The L-function of an automorphic form

We have shown in theorem 2.1.9 that every automorphic form ϕ on $GL(1, \mathbb{A}_{\mathbb{Q}})$ is associated to a uniquely defined Dirichlet character χ. To each such automorphic form ϕ we shall attach an L-function:

$$L(s, \phi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \tag{2.2.1}$$

where the above product converges absolutely for $\Re(s) > 1$. The L-function (2.2.1) is just the classical Dirichlet L-function associated to the Dirichlet character χ.

What we have done here is to take some classical objects:

Dirichlet character, Dirichlet L-function,

and dress them in very fancy clothes so that they become almost unrecognizable extremely elegant objects with a new personality. Nevertheless, we are undaunted because we know them for what they truly are. However, there is much to be gained by such an approach and the later rewards will be very significant.

Our next goal is to obtain the analytic continuation and functional equation of $L(s, \phi)$ by the adelic method of [Tate, 1950], [Iwasawa, 1952, 1992]. In general, the L-function (2.2.1) can be constructed by integrating (over the idele group) the automorphic form ϕ against a suitable test function. Let us begin by discussing the simplest case when the automorphic form is the trivial function, i.e., the constant one. In this case, the L-function is just the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

In order to construct the Riemann zeta function as an idelic integral, we must introduce a set of test functions and appropriate measures to do idelic integration. Here are the precise definitions we need.

Definition 2.2.2 (Adelic Schwartz space) *Let $\mathbb{S}$ denote the set of all adelic Schwartz functions as defined in definition 1.7.4. Recall that these are just linear combinations of functions $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ where $f = \prod\limits_{v \le \infty} f_v$ (with f_v the characteristic function of $\mathbb{Z}_v$ for all but finitely many $v \le \infty$) and f_∞ is Schwartz on $\mathbb{R}$ and f_v is a locally constant compactly supported function at all finite primes v.*

Definition 2.2.3 (Idelic integral) *As in definition 1.7.5, we define the idelic integral for factorizable idelic functions $f = \prod_v f_v$ such that f_p is the characteristic function $1_{\mathbb{Z}_p^\times}$ for almost all primes p by*

$$\int_{\mathbb{A}_{\mathbb{Q}}^\times} f(x) d^\times x \; = \; \prod_{v \in S} \int_{\mathbb{Q}_v^\times} f_v(x_v) d^\times x_v,$$

where S is a finite set containing ∞ and all the primes such that $f_p \ne 1_{\mathbb{Z}_p^\times}$. Also,

$$d^\times x_v = \begin{cases} \frac{dx_\infty}{|x_\infty|_\infty}, & \text{if } v = \infty, \\ \frac{1}{1-p^{-1}} \frac{dx_p}{|x_p|_p}, & \text{if } v = p \text{ is a finite prime.} \end{cases}$$

Thus, $d^\times x_p$ is normalized so that $\int_{\mathbb{Z}_p^\times} d^\times x_p = 1$. In the usual manner, we extend the definition of adelic integration to define $\int_E f(x) d^\times x$ for functions $f : \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}$, and subsets E of $\mathbb{A}_{\mathbb{Q}}^\times$ such that $f \cdot 1_E$ is a linear combination of factorizable functions of the type considered above. As in the additive theory, we have

$$\int_{\mathbb{A}_{\mathbb{Q}}^\times} \prod_{v \le \infty} f_v(x_v)\, d^\times x = \prod_{v \le \infty} \int_{\mathbb{Q}_v^\times} f_v(x_v)\, d^\times x_v.$$

We summarize this information as defining an idelic differential:

$$d^\times x = \prod_{v \le \infty} d^\times x_v.$$

Definition 2.2.4 (Idelic absolute value) *For $x = \{x_\infty, x_2, \ldots\} \in \mathbb{A}_{\mathbb{Q}}^\times$, we recall the definition*

$$|x|_{\mathbb{A}} = \prod_{v \le \infty} |x_v|_v.$$

Definition 2.2.5 (Special choice of test function) *For $x = \{x_\infty, x_2, \ldots\} \in \mathbb{A}_{\mathbb{Q}}^\times$, we define the test function*

$$h(x) = e^{-\pi x_\infty^2} \prod_{v < \infty} 1_{\mathbb{Z}_v}(x_v) \ \in \ \mathbb{S}$$

where $1_{\mathbb{Z}_v}$ is the characteristic function of $\mathbb{Z}_v$ at all the finite primes $v < \infty$. The function h has the nice property that $h = \hat{h}$, i.e., it is its own Fourier transform.

Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Following example 1.5.8, the Riemann zeta function, $\zeta(s)$, then appears naturally in the following computation:

$$\begin{aligned} \int_{\mathbb{A}_{\mathbb{Q}}^\times} h(x) |x|_{\mathbb{A}}^s \, d^\times x \ &= \ \int_{\mathbb{R}^\times} e^{-\pi t^2} \, |t|_\infty^s \, \frac{dt}{|t|_\infty} \ \cdot \ \prod_p \int\limits_{\mathbb{Z}_p - \{0\}} |x_p|_p^s \, d^\times x_p \\ &= \pi^{-\frac{s}{2}} \, \Gamma\left(\frac{s}{2}\right) \ \cdot \ \prod_p \left(1 - p^{-s}\right)^{-1} \\ &= \pi^{-\frac{s}{2}} \, \Gamma\left(\frac{s}{2}\right) \zeta(s). \end{aligned} \tag{2.2.6}$$

The meromorphic continuation and functional equation of $\zeta(s)$ can be obtained by use of the adelic Poisson summation formula (1.9.2). Since $h = \hat{h}$, and more specifically, $\hat{h}(0) = h(0) = 1$, we may rewrite the adelic Poisson summation formula (1.9.2) in the form:

$$1 + \sum_{\alpha \in \mathbb{Q}^\times} h(\alpha x) = \frac{1}{|x|_{\mathbb{A}}} \ + \ \frac{1}{|x|_{\mathbb{A}}} \sum_{\alpha \in \mathbb{Q}^\times} h\left(\frac{\alpha}{x}\right). \tag{2.2.7}$$

Recall proposition 1.4.6 which says that

$$\mathbb{A}_\mathbb{Q}^\times = \bigcup_{\alpha\in\mathbb{Q}^\times} \alpha\cdot\left(\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\right).$$

To obtain the analytic continuation and functional equation of $\zeta(s)$ we proceed as follows:

$$\begin{aligned}
\int_{\mathbb{A}_\mathbb{Q}^\times} h(x)|x|_\mathbb{A}^s\, d^\times x &= \sum_{\alpha\in\mathbb{Q}^\times}\ \int\limits_{\alpha\cdot\left(\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\right)} h(x)\,|x|_\mathbb{A}^s\, d^\times x\\
&= \int\limits_{\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times}\ \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\,|\alpha x|_\mathbb{A}^s\, d^\times x
\end{aligned}$$

(2.2.8)

$$= \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}}\ \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\,|x|_\mathbb{A}^s\, d^\times x + \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\ge 1}}\ \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\,|x|_\mathbb{A}^s\, d^\times x.$$

In the above we have used the product formula (theorem 1.1.8) which says that $|\alpha|_\mathbb{A} = 1$ for $\alpha\in\mathbb{Q}^\times$. We apply the Poisson summation formula (2.2.7) to the first term in the last line of (2.2.8) to obtain

(2.2.9)

$$\begin{aligned}
\int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}}\ \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\,|x|_\mathbb{A}^s\, d^\times x &= \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}} \left(\frac{1}{|x|_\mathbb{A}} + \frac{1}{|x|_\mathbb{A}}\sum_{\alpha\in\mathbb{Q}^\times} h\left(\frac{\alpha}{x}\right) - 1\right)|x|_\mathbb{A}^s\, d^\times x\\
&= \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}} \left(|x|_\mathbb{A}^{s-1} - |x|_\mathbb{A}^s\right) d^\times x + \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}}\ \sum_{\alpha\in\mathbb{Q}^\times} h\left(\frac{\alpha}{x}\right)|x|_\mathbb{A}^{s-1}\, d^\times x\\
&= \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}} \left(|x|_\mathbb{A}^{s-1} - |x|_\mathbb{A}^s\right) d^\times x + \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\ge 1}}\ \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)|x|_\mathbb{A}^{1-s}\, d^\times x.
\end{aligned}$$

By proposition 1.4.5 (strong approximation for ideles) we have

$$\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times = (0,\infty)\cdot\prod_p \mathbb{Z}_p^\times.$$

It follows that for $\Re(s) > 0$,

$$\int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\le 1}} |x|_\mathbb{A}^s\, d^\times x = \int_0^1 y^s\,\frac{dy}{y} = \frac{1}{s}.$$

Combining the above computations with (2.2.6), (2.2.8), (2.2.9) yields

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int\limits_{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times} \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\,|x|_\mathbb{A}^s\,d^\times x$$

$$= \frac{1}{s-1} - \frac{1}{s} + \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\ge 1}} \sum_{\alpha\in\mathbb{Q}^\times} h(\alpha x)\Big(|x|_\mathbb{A}^s + |x|_\mathbb{A}^{1-s}\Big)\,d^\times x. \tag{2.2.10}$$

Note that the integral on the right side of (2.2.10) converges for all $s\in\mathbb{C}$ and defines an entire function. Consider x on the right side of (2.2.10). Since x is restricted to be in the fundamental domain for $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$, it follows that $x_p\in\mathbb{Z}_p^\times$ for all finite primes p. It further follows from the definition of h that $\alpha x_p\in\mathbb{Z}_p$ for all finite primes p. This implies that $\alpha\in\mathbb{Z}$. After some elementary manipulations it is easy to show that equation (2.2.10) is really the same as the classical identity

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2y^2}\,|y|_\infty^s\,\frac{dy}{y}$$

$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2y^2}\left(|y|_\infty^s + |y|_\infty^{1-s}\right)\frac{dy}{y}. \tag{2.2.11}$$

One immediately obtains the following theorem.

Theorem 2.2.12 *The Riemann zeta function,* $\zeta(s)=\sum_{n=1}^\infty n^{-s}$, *has a meromorphic continuation to all* $s\in\mathbb{C}$ *with a simple pole at* $s=1$. *It satisfies the functional equation*

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) \;=\; \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Tate, in his thesis [Tate, 1950] realized that an identity of type 2.2.10 could be obtained for any test function f in the adelic Schwartz space $\mathbb{S}$ defined in 2.2.2. If we rewrite the adelic Poisson summation formula (1.9.2) in the form

$$\sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x) = \frac{\hat f(0)}{|x|_\mathbb{A}} + \frac{1}{|x|_\mathbb{A}}\sum_{\alpha\in\mathbb{Q}^\times}\hat f\left(\frac{\alpha}{x}\right) - f(0),$$

and we replicate the steps in the proof of (2.2.10), using a more general function f instead of h, it immediately follows that

(2.2.13)

$$\int\limits_{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times} \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\,|x|_\mathbb{A}^s\,d^\times x$$

$$= \frac{\hat f(0)}{s-1} - \frac{f(0)}{s} + \int\limits_{\substack{x\in\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times\\ |x|_\mathbb{A}\ge 1}} \sum_{\alpha\in\mathbb{Q}^\times}\Big(f(\alpha x)|x|_\mathbb{A}^s + \hat f(\alpha x)|x|_\mathbb{A}^{1-s}\Big)\,d^\times x.$$

Note that the right side of (2.2.13) does not change if we make the simultaneous transformations:

$$s \to 1-s, \qquad f \to \hat{f}.$$

Now, let us assume that $f \in \mathbb{S}$ is factorizable, as in definition 1.7.3. Then f_p is the characteristic function of $\mathbb{Z}_p$ for all but finitely many primes p. Let $S = \{p_1, p_2, \ldots, p_\ell\}$ denote the finite set of primes p_i where f_{p_i} is not equal to the characteristic function of $\mathbb{Z}_{p_i} (i = 1, 2, \ldots, \ell)$.

It follows that

$$\begin{aligned} \int_{\mathbb{A}_{\mathbb{Q}}^\times} f(x)|x|_{\mathbb{A}}^s \, d^\times x &= \int_{\mathbb{R}^\times} f_\infty(y)\,|y|_\infty^s \,\frac{dy}{|y|_\infty} \cdot \prod_p \int\limits_{\mathbb{Q}_p^\times} f_p(x_p)\,|x_p|_p^s \, d^\times x_p \\ &= \tilde{f}_\infty(s) \cdot \left(\prod_{p \in S} \int\limits_{\mathbb{Q}_p^\times} f_p(x_p)\,|x_p|_p^s \, d^\times x_p \right) \cdot \prod_{p \notin S} \left(1 - p^{-s}\right)^{-1} \qquad (2.2.14) \\ &= \tilde{f}_\infty(s) \cdot \left(\prod_{p \in S} \frac{\int\limits_{\mathbb{Q}_p^\times} f_p(x_p)\,|x_p|_p^s \, d^\times x_p}{\left(1 - p^{-s}\right)^{-1}} \right) \zeta(s), \end{aligned}$$

where $\tilde{f}$ denotes the Mellin transform of f given by

$$\tilde{f}(s) = \int_0^\infty f(y) y^s \frac{dy}{y}.$$

We immediately conclude that the right hand side of (2.2.14) is invariant under the simultaneous transformations

$$\boxed{s \to 1-s, \qquad f \to \hat{f}.}$$

INTERLUDE (Three remarks of Ivan Fesenko):

Remark (1): *Equations (2.2.13) and (2.2.14) are further generalized in Tate's thesis* [Tate, 1950]. *The method is a very powerful and amazingly simple way to deduce the key properties of zeta and L-functions. Exactly the same reasoning as above gives the functional equation and meromorphic continuation of the completed zeta function of an algebraic number field K. The analogue of the right hand side of (2.2.10) is*

$$c\left(\frac{1}{s-1} - \frac{1}{s}\right) + \int\limits_{\substack{x \in K^\times \backslash \mathbb{A}_K^\times \\ |x|_{\mathbb{A}} \geq 1}} \sum_{\alpha \in K^\times} h(\alpha x) \left(|x|_{\mathbb{A}}^s + |x|_{\mathbb{A}}^{1-s}\right) \, d^\times x$$

where c is the volume of $K^\times \backslash \mathbb{A}_K'$, and $\mathbb{A}_K' = \{x \in \mathbb{A}_K^\times \mid |x|_{\mathbb{A}} = 1\}$.

As first observed in [Iwasawa, 1952]*, the above formula implies the finiteness of* c *which in turn implies the compactness of* $K^\times \backslash \mathbb{A}'_K$ *and the finiteness of the class group of* K*. It also easily implies the Dirichlet theorem of units in* K*. So, the most fundamental theorems of classical algebraic number theory follow as easy and fast corallaries of the adelic computation of the zeta function.*

Remark (2): *Ideles were first introduced by Chevalley in the 1930's to simplify and make more conceptual the exposition of class field theory. The adelic method of Tate and Iwasawa works with functions on adelic objects but does not use any result of class field theory. An extension of the adelic method from* $GL(1)$ *to* $GL(n)$ *was first found by Godement and Jacquet (see* [Godement-Jacquet, 1972]*) and is a major theme of this book. The extension gives a noncommutative theory over global fields, which is related to noncommutative class field theory and the Langlands program. The latter is about 40 years old and is still actively involved in the discovery of new fundamental concepts.*

Remark (3): *Instead of going from* $GL(1)$ *to* $GL(n)$*, it is also possible to extend the work of Tate and Iwasawa in an entirely different "aspect." From the point of view of modern algebraic geometry, the field* $\mathbb{Q}$ *may be thought of as the field of rational functions on the one dimensional arithmetic scheme Spec*$(\mathbb{Z})$*. A two dimensional adelic analysis seeks to develop and use harmonic analysis on adelic spaces associated to an arithmetic surface (i.e., a two dimensional object). For example, to an elliptic curve* E *over a global field* K*, given by equation* $Y^2 = X^3 + aX + b$*, one can associate an arithmetic surface* $\mathcal{E}$*.*

Two dimensional (abelian) class field theory [Parshin, Kato, Saito, Fesenko] *(and others) describes abelian extensions of the field of rational functions on* $\mathcal{E}$ *via certain adelic objects and their quotients. The adelic objects are restricted products of certain two-dimensional local fields, e.g.,* $\mathbb{Q}_p((t)), \mathbb{R}((t))$*, etc., which are not locally compact groups. See* [Fesenko, 2003, 2008] *for a development of a two-dimensional adelic theory and a generalization of (2.2.10) in this context. The two-dimensional adelic theory studies a zeta integral which is closely related to the arithmetic zeta function of* $\mathcal{E}$ *which itself is essentially* $\zeta_K(s)$ *times* $\zeta_K(s-1)$ *divided by the Hasse-Weil L-function of* E*. Note that, as in the classical, one-dimensional case, the study of zeta functions uses adelic objects which naturally show up in class field theory, but does not depend on results from class field theory.*

We shall now show how the adelic method of Tate and Iwasawa can be used to obtain the meromorphic continuation and functional equation of all $GL(1, \mathbb{A}_{\mathbb{Q}})$ L-functions.

Let ϕ be a non-zero automorphic form for $GL(1, \mathbb{A}_{\mathbb{Q}})$ as in definition 2.1.4. By theorem 2.1.9, the automorphic form ϕ is determined by a constant $c \neq 0$, a real number t, and the idelic lift χ_{idelic} of a Dirichlet character χ. Let $f \in \mathbb{S}$ be an adelic Schwartz function as in definition 2.2.2. For $s \in \mathbb{C}$ with $\Re(s) > 1$, the Dirichlet L-function (2.2.1) associated to ϕ will appear naturally in the computation of the adelic integral

$$(2.2.15) \qquad \int_{\mathbb{A}_{\mathbb{Q}}^\times} f(x)\, \chi_{\text{idelic}}(x)\, |x|_{\mathbb{A}}^s \, d^\times x \; = \; c^{-1} \int_{\mathbb{A}_{\mathbb{Q}}^\times} f(x)\, \phi(x)\, |x|_{\mathbb{A}}^{s-it} \, d^\times x.$$

Let $S = \{p_1, p_2, \ldots, p_\ell\}$ be the finite set of primes where χ_p is ramified (see definition 2.1.14). Then χ_p is unramified for $p \notin S$. We shall now also make the assumption that f_p is equal to the characteristic function of $\mathbb{Z}_p$ for $p \notin S$. The computation of (2.2.14) easily generalizes to the situation of (2.2.15). We have

(2.2.16)

$$\begin{aligned}
&\int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x)\,\chi_{\text{idelic}}(x)\,|x|_{\mathbb{A}}^{s}\,d^{\times}x \\
&\quad= \int_{\mathbb{R}^{\times}} f_\infty(y)\,\chi_\infty(y)\,|y|_\infty^{s}\,\frac{dy}{|y|_\infty}\cdot\prod_p \int\limits_{\mathbb{Q}_p^{\times}} f_p(x_p)\,\chi_p(x_p)\,|x_p|_p^{s}\,d^{\times}x_p \\
&\quad= \widetilde{f_\infty\chi_\infty}(s)\cdot\left(\prod_{p\in S}\int\limits_{\mathbb{Q}_p^{\times}} f_p(x_p)\,\chi_p(x_p)\,|x_p|_p^{s}\,d^{\times}x_p\right)\cdot\prod_{p\notin S}\left(1-\chi(p)p^{-s}\right)^{-1} \\
&\quad= \widetilde{f_\infty\chi_\infty}(s)\cdot\left(\prod_{p\in S}\frac{\int\limits_{\mathbb{Q}_p^{\times}} f_p(x_p)\,\chi_p(x_p)\,|x_p|_p^{s}\,d^{\times}x_p}{\left(1-\chi(p)p^{-s}\right)^{-1}}\right)L(s,\chi) \\
&\quad= \widetilde{f_\infty\chi_\infty}(s)\cdot\left(\prod_{p\in S}\int\limits_{\mathbb{Q}_p^{\times}} f_p(x_p)\,\chi_p(x_p)\,|x_p|_p^{s}\,d^{\times}x_p\right)L(s,\chi)
\end{aligned}$$

where

$$\widetilde{f_\infty\chi_\infty}(s) = \int_{\mathbb{R}^{\times}} f_\infty(y)\,\chi_\infty(y)\,|y|_\infty^{s}\,\frac{dy}{|y|_\infty}$$

denotes the Mellin transform of $f_\infty\chi_\infty$. Note, also, that if χ_p is ramified then $\chi(p) = 0$. This fact is used to obtain the last line in (2.2.16).

Furthermore, equation (2.2.13) also easily generalizes to this situation and we may obtain

(2.2.17)

$$\begin{aligned}
&\int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x)\,\chi_{\text{idelic}}(x)\,|x|_{\mathbb{A}}^{s}\,d^{\times}x \\
&\qquad= \int\limits_{x\in\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times}}\ \sum_{\alpha\in\mathbb{Q}^{\times}} f(\alpha x)\chi_{\text{idelic}}(\alpha x)\,|x|_{\mathbb{A}}^{s}\,d^{\times}x \\
&\qquad= \int\limits_{\substack{x\in\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times}\\ |x|_{\mathbb{A}}\le 1}} \left(\hat{f}(0)\,|x|_{\mathbb{A}}^{s-1} \;-\; f(0)\,|x|_{\mathbb{A}}^{s}\right)\chi_{\text{idelic}}(x)\,d^{\times}x \\
&\qquad\qquad+ \int\limits_{\substack{x\in\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times}\\ |x|_{\mathbb{A}}\ge 1}}\ \sum_{\alpha\in\mathbb{Q}^{\times}}\left(f(\alpha x)\chi_{\text{idelic}}(x)|x|_{\mathbb{A}}^{s} + \hat{f}(\alpha x)\overline{\chi_{\text{idelic}}(x)}|x|_{\mathbb{A}}^{1-s}\right)d^{\times}x.
\end{aligned}$$

In the last line above we have used the fact that

$$\chi_{\text{idelic}}(1/x) = \overline{\chi_{\text{idelic}}(x)},$$

which holds because χ_{idelic} is unitary, i.e., has absolute value $= 1$. Note that the last integral in (2.2.17) converges absolutely for all $s \in \mathbb{C}$ and defines an entire function. It is clear that the right most integral on the right side of equation (2.2.17) is invariant under the simultaneous transformations:

$$s \to 1 - s \qquad f \to \hat{f}, \qquad \chi_{\text{idelic}} \to \overline{\chi_{\text{idelic}}}. \tag{2.2.18}$$

One may also show that

$$\int\limits_{\substack{x \in \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \\ |x|_\mathbb{A} \le 1}} \Big(\hat{f}(0) \overline{\chi_{\text{idelic}}(x)} |x|_\mathbb{A}^{s-1} \; - \; f(0) \chi_{\text{idelic}}(x) |x|_\mathbb{A}^s \Big) \, d^\times x = 0 \tag{2.2.19}$$

if χ_{idelic} has ramification (as in definition 2.1.14) at some finite prime.

The fact that the left hand side of (2.2.19) vanishes is essentially due to the fact that a non-trivial Dirichlet character $\chi \pmod q$ satisfies

$$\sum_{\substack{j=1 \\ (j,q)=1}}^{q} \chi(j) = 0.$$

It follows from (2.2.17) that if χ_{idelic} has ramification, then the L-function $L(s, \chi)$ has a holomorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation:

$$s \to 1 - s.$$

Note that the holomorphy follows since there is a particular choice of f in (2.2.17) for which $\hat{f}_\infty$ is everywhere nonvanishing. If χ_{idelic} is everywhere unramified, then we are in the situation of the Riemann zeta function and the formulas take the form of (2.2.9).

Let χ be a primitive Dirichlet character $\pmod q$. It is known (see [Davenport, 2000]) that the Dirichlet L-function $L(s, \chi)$ has holomorphic continuation to the entire complex plane and satisfies the explicit functional equation

$$\boxed{\xi^*(s, \chi) := \left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\mathfrak{a})} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s, \chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}} \sqrt{q}} \, \xi^*(1 - s, \bar{\chi})} \tag{2.2.20}$$

where

$$\tau(\chi) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(a) e^{\frac{2\pi i a}{q}}$$

is the Gauss sum and

$$\mathfrak{a} = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

Definition 2.2.21 (Global conductor and root number) *The integer q in (2.2.20) is called the conductor of $L(s,\chi)$ while the constant $\frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}$ is called the root number. Tate defined a different root number (which depends on s and is not of absolute value one). Tate's root number is defined to be $\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \cdot \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}$.*

Tate realized that the identities (2.2.16) and (2.2.17) which are invariant under the transformations (2.2.18) must encode the classical functional equation (2.2.20) of the Dirichlet L-function. The fact that (2.2.16) and (2.2.17) hold for any adelic Schwartz function $f \in \mathbb{S}$ suggests that there must be further symmetries hidden in these identities. This led him to discover the so called local functional equations and the fact that the analytic conductor and root number can be determined by local conditions. In the next two sections we shall introduce the local L-functions and their functional equations and explain how the global functional equation (2.2.20) follows from them.

§2.3 The local L-functions and their functional equations

In this section let $v \le \infty$ determine a local field $\mathbb{Q}_v$.

Definition 2.3.3 (Local L-function) *Fix $s \in \mathbb{C}$ with $\Re(s) > 0$. Let $f_v : \mathbb{Q}_v \to \mathbb{C}$ be a locally constant compactly supported function as in definition 1.5.2 if $v < \infty$ and a Schwartz function if $v = \infty$. Let $\omega_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ be a local unitary character, i.e., a continuous homomorphism of absolute value 1. We define*

$$L(s, f_v, \omega_v) = \int_{\mathbb{Q}_v^\times} f_v(x)\, \omega_v(x)\, |x|_v^s \, d^\times x, \tag{2.3.4}$$

to be the local L-function associated to ω_v and f_v. Here $d^\times x$ denotes the differential associated to the multiplicative Haar measure as in definition 1.5.7 if $v < \infty$ and $d^\times x = dx/|x|_\infty$ if $v = \infty$.

Remark 2.3.5: The local integral (2.3.4) is absolutely convergent if $\Re(s) > 0$. To see this we write

$$L(s, f_v, \omega_v) = \int\limits_{\substack{x \in \mathbb{Q}_v^\times \\ |x|_v > 1}} f_v(x)\, \omega_v(x)\, |x|_v^s \, d^\times x + \int\limits_{\substack{x \in \mathbb{Q}_v^\times \\ |x|_v \le 1}} f_v(x)\, \omega_v(x)\, |x|_v^s \, d^\times x.$$

The first integral on the right above is clearly convergent. Since f_v is bounded in the region $|x|_v \le 1$, the second integral is bounded by

$$\int\limits_{\substack{x \in \mathbb{Q}_v^\times \\ |x|_v \le 1}} |x|_v^\sigma \, d^\times x,$$

an integral which converges absolutely for $\Re(s) = \sigma > 0$ for any fixed $v \le \infty$.

We now state the main theorem in the local theory.

Theorem 2.3.6 (Local functional equation) *Let $s \in \mathbb{C}$ with $0 < \Re(s) < 1$. The local L-function $L(s,\, f_v,\, \omega_v)$, as defined in 2.3.3, satisfies the functional equation*

$$L(s,\, f_v,\, \omega_v) = \rho_{\omega_v}(s) \cdot L(1-s,\, \hat{f}_v,\, \overline{\omega_v})$$

where $\hat{f}_v$ denotes the v-adic Fourier transform of f_v (see 1.6.1, 1.6.8), $\overline{\omega_v} = \omega_v^{-1}$ is the complex conjugate of ω_v, and $\rho_{\omega_v}(s)$ is a meromorphic function which is independent of the choice of f_v.

Proof: To show that ρ_{ω_v} is independent of f_v it is enough to show that

$$\frac{L(s,\, f_v,\, \omega_v)}{L(1-s,\, \hat{f}_v,\, \overline{\omega_v})} = \frac{L(s,\, g_v,\, \omega_v)}{L(1-s,\, \hat{g}_v,\, \overline{\omega_v})}$$

for any function $g_v : \mathbb{Q}_v \to \mathbb{C}$ with the same properties as f_v. We will prove that

$$L(s,\, f_v,\, \omega_v)L(1-s,\, \hat{g}_v,\, \overline{\omega_v}) = L(1-s,\, \hat{f}_v,\, \overline{\omega_v})L(s,\, g_v,\, \omega_v)$$

by showing that

$$L(s,\, f_v,\, \omega_v)L(1-s,\, \hat{g}_v,\, \overline{\omega_v}) \tag{2.3.7}$$

is symmetric in f_v and g_v. By remark 2.3.5, if $0 < Re(s) < 1$, we can write (2.3.7) as an absolutely convergent double integral

$$\int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v^\times} f_v(x)\hat{g}_v(y)\omega_v\left(xy^{-1}\right) |x|_v^s\, |y|_v^{1-s}\, d^\times x\, d^\times y. \tag{2.3.8}$$

The measure $d^\times y$ is invariant under the transformation $y \to xy$. It follows that (2.3.8) can be rewritten in the form

$$\int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v^\times} f_v(x)\hat{g}_v(xy)\overline{\omega\,(y)}|x|_v\, |y|_v^{1-s}\, d^\times x\, d^\times y.$$

But

$$\hat{g}_v(xy) = \int_{\mathbb{Q}_v} g_v(z)e_v(-xyz)dz.$$

Consequently

$$\begin{aligned}
&\int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v^\times} f_v(x)\hat{g}_v(xy)\overline{\omega\,(y)}|x|_v\, |y|_v^{1-s}\, d^\times x\, d^\times y \\
&\quad = \int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v^\times} \int_{\mathbb{Q}_v} f_v(x)g_v(z)e_v(-xyz)\overline{\omega\,(y)}|y|_v^{1-s}\, dx\, d^\times y\, dz \cdot \begin{cases} \frac{v}{v-1}, & \text{if } v \neq \infty, \\ 1, & \text{if } v = \infty, \end{cases}
\end{aligned}$$

which is clearly symmetric in f_v and g_v.

□

Proposition 2.3.9 *Let $s \in \mathbb{C}$ with $0 < \Re(s) < 1$. Let $\rho_{\omega_v}(s)$ be defined as in theorem 2.3.6. Then*

$$\rho_{\omega_v}(s) \cdot \rho_{\overline{\omega}_v}(1-s) = 1.$$

Proof: By definition

$$L(s, f_v, \omega_v) = \rho_{\omega_v}(s) L(1-s, \hat{f}_v, \overline{\omega}_v),$$

$$L(1-s, \hat{f}_v, \overline{\omega}_v) = \rho_{\overline{\omega}_v}(1-s) L(s, f, \omega_v).$$

Consequently

$$L(s, f, \omega_v) = \rho_{\omega_v}(s)\rho_{\overline{\omega}_v}(1-s) L(s, f_v, \omega_v),$$

which implies that

$$\rho_{\omega_v}(s) \cdot \rho_{\overline{\omega}_v}(1-s) = 1.$$

□

Theorem 2.3.10 (Explicit computation of $\rho_{\omega_v}(s)$) *Let $s \in \mathbb{C}$, $0 < \Re(s) < 1$. The function $\rho_{\omega_v}(s)$ can be explicitly given as follows.*

Case 1: $v = \infty$

$$\rho_{\omega_\infty}(s) = \begin{cases} \dfrac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}, & \text{if } \omega_\infty(x) = 1 \text{ for } x \in \mathbb{R}, \\[2ex] \frac{1}{i}\cdot \dfrac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)}, & \text{if } \omega_\infty(x) = \operatorname{sign}(x) \text{ for } x \in \mathbb{R}. \end{cases}$$

Case 2: $v = p < \infty$

$$\rho_{\omega_p}(s) = \begin{cases} \left(1 - \frac{\overline{\omega_p(p)}}{p^{1-s}}\right) \Big/ \left(1 - \frac{\omega_p(p)}{p^s}\right), & \text{if } \omega_p \text{ is unramified}, \\[2ex] \frac{p^{r(s-1)}}{\omega_p(p)^r} \sum\limits_{\substack{j=1 \\ (j,p)=1}}^{p^r} \omega_p(j) e^{-2\pi i j p^{-r}}, & \omega_p \text{ is ramified and has conductor } p^r. \end{cases}$$

Remarks: The local L-function $L(s, f_v, \omega_v)$ can easily be $= 0$ if the Schwartz function f_v is chosen in a stupid way. In the proof below we have made smart choices $f_v = f_v^\circ$ in each situation. The factor $1/i$ that appears in $\rho_{\omega_\infty}(s)$ when $\omega_\infty(x) = \operatorname{sign}(x)$, occurs in the global root number, of (2.2.20) for the case when $\chi(-1) = -1$. When ω_p is ramified and has conductor p^r then $\rho_{\omega_p}(s)$ contains the Gauss sum

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{p^r} \omega_p(j) e^{-2\pi i \frac{j}{p^r}},$$

and when these Gauss sums are multiplied together over all ramified primes we obtain the Gauss sum occurring in the root number (2.2.20).

Proof of theorem 2.3.10: We first consider the case $v = \infty$, $f_\infty^\circ(x) = e^{-\pi x^2}$, and $\omega_\infty \equiv 1$. Then the local L-function is

$$L(s, f_\infty^\circ, \omega_\infty) = \int_{-\infty}^{\infty} e^{-\pi x^2} |x|_\infty^s \, \frac{dx}{|x|_\infty} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \tag{2.3.11}$$

Since $f_\infty^\circ = \hat{f}_\infty^\circ$ and $\overline{\omega}_\infty = \omega_\infty$, it follows that

$$\rho_{\omega_\infty}(s) = \frac{L(s, f_\infty^\circ, \omega_\infty)}{L(1-s, f_\infty^\circ, \omega_\infty)} = \frac{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)}.$$

Next, consider the case $v = \infty$, $f_\infty^\circ(x) = x e^{-\pi x^2}$, and $\omega_\infty(x) = \text{sign}(x) = \frac{|x|_\infty}{x}$. Consequently

$$L(s, f_\infty^\circ, \omega_\infty) = \int_{-\infty}^{\infty} e^{-\pi x^2} |x|_\infty^{s+1} \, \frac{dx}{|x|_\infty} = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right). \tag{2.3.12}$$

A simple computation shows that $\widehat{f_\infty^\circ}(x) = \int_{-\infty}^{\infty} f_\infty^\circ(y) e^{2\pi i x y} dy = i f_\infty^\circ(x)$. Hence

$$L(1-s, \widehat{f_\infty^\circ}, \overline{\omega}) = i \int_{-\infty}^{\infty} e^{-\pi x^2} |x|_\infty^{(1-s)+1} \, \frac{dx}{|x|_\infty} = i \pi^{-\frac{(1-s)+1}{2}} \Gamma\left(\frac{(1-s)+1}{2}\right).$$

It follows that

$$\rho_{\omega_\infty}(s) = \frac{L(s, f_\infty^\circ, \omega_\infty)}{L(1-s, \widehat{f_\infty^\circ}, \overline{\omega}_\infty)} = \frac{1}{i} \, \frac{\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}} \Gamma\left(\frac{(1-s)+1}{2}\right)}.$$

We now consider the non-archimedean situation where $v = p$ is a finite prime and ω_p is unramified. Choose

$$f_p^\circ(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} L(s, f_p^\circ, \omega_p) &= \frac{p}{p-1} \int_{\mathbb{Q}_p^\times} f_p^\circ(x) \omega_p(x) |x|_p^s \, \frac{dx}{|x|_p} = \frac{p}{p-1} \int_{\mathbb{Z}_p - \{0\}} \omega_p(x) |x|_p^s \, \frac{dx}{|x|_p} \\ &= 1 + \frac{\omega_p(p)}{p^s} + \frac{\omega_p(p^2)}{p^{2s}} + \frac{\omega_p(p^3)}{p^{3s}} + \cdots \\ &= \left(1 - \frac{\omega_p(p)}{p^s}\right)^{-1}. \end{aligned} \tag{2.3.13}$$

Since $\widehat{f_p^\circ} = f_p^\circ$, we also have

$$L(1-s, \widehat{f_p^\circ}, \overline{\omega}) = \left(1 - \frac{\overline{\omega_p(p)}}{p^{1-s}}\right)^{-1}.$$

Consequently

$$\rho_{\omega_p}(s) = \frac{\left(1 - \frac{\overline{\omega_p(p)}}{p^{1-s}}\right)}{\left(1 - \frac{\omega_p(p)}{p^s}\right)}.$$

Finally, we consider $v = p$ and ω_p is ramified with conductor p^r as in definition (2.1.14). In this case, we choose

$$f_p^\circ(x) = \begin{cases} e^{-2\pi i\{x\}}, & \text{if } x \in p^{-r}\mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{Q}_p^\times$ as in definition 1.6.3. With these choices we obtain

$$\begin{aligned} L(s, f_p^\circ, \omega_p) = &\ \frac{p}{p-1} \int\limits_{p^{-f}\mathbb{Z}_p - \{0\}} e^{-2\pi i\{x\}}\omega_p(x)\,|x|_p^s\,\frac{dx}{|x|_p} \\ = &\ \frac{p}{p-1}\sum_{\ell=1}^{r}\sum_{\substack{j=1\\(j,p)=1}}^{p^r} \int\limits_{p^{-\ell}(j+p^r\mathbb{Z}_p)} e^{-2\pi i\{x\}}\,\omega_p(x)\,|x|_p^s\,\frac{dx}{|x|_p} \qquad (2.3.14) \\ = &\ p^{-r}\frac{p}{p-1}\cdot\sum_{\ell=1}^{r} p^{\ell s}\,\omega_p(p)^{-\ell}\sum_{\substack{j=1\\(j,p)=1}}^{p^r} e^{\frac{-2\pi i j}{p^\ell}}\omega_p(j) \\ = &\ \frac{p^{-r+1}}{p-1}\cdot p^{rs}\omega_p(p)^{-r}\cdot\sum_{j=1}^{p^r} e^{-\frac{2\pi i j}{p^r}}\omega_p(j). \end{aligned}$$

In the above computation, we used the fact that

$$\sum_{\substack{j=1\\(j,p)=1}}^{p^r} e^{\frac{-2\pi i j}{p^\ell}}\omega_p(j) = 0$$

if $\ell < r$.

In order to compute $L(1-s, \widehat{f_p^\circ}, \overline{\omega_p})$ we need to compute the Fourier transform $\widehat{f_p^\circ}$. We have

$$\begin{aligned} \widehat{f_p^\circ}(x) &= \int\limits_{\mathbb{Q}_p} f_p^\circ(y)e^{2\pi i\{xy\}}\,dy \\ &= \int\limits_{p^{-r}\mathbb{Z}_p} e^{-2\pi i\{y(1-x)\}}\,dy \\ &= \begin{cases} p^r, & \text{if } x \in 1 + p^r\mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} L(1-s, \widehat{f_p^\circ}, \overline{\omega_p}) &= p^r \frac{p}{p-1} \int\limits_{1+p^r\mathbb{Z}_p} \overline{\omega_p(x)}\, |x|_p^{1-s} \, \frac{dx}{|x|_p} \\ &= p^r \frac{p}{p-1} \int\limits_{1+p^r\mathbb{Z}_p} \frac{dx}{|x|_p} \\ &= \frac{p}{p-1}. \end{aligned}$$

□

Theorem 2.3.10 can be used to show that the local L-functions, as defined in 2.3.3, have meromorphic continuation to the whole complex plane.

Theorem 2.3.15 (Meromorphic continuation of the local L-functions) *Fix $s \in \mathbb{C}$ with $\Re(s) > 0$. Let $L(s, f_v, \omega_v)$ denote the local L-function as in definition 2.3.3. Then $L(s, f_v, \omega_v)$ has meromorphic continuation to all $s \in \mathbb{C}$.*

Proof: The explicit computation of $\rho_{\omega_v}(s)$ given in theorem 2.3.10 shows that in all cases, $\rho_{\omega_v}(s)$ has meromorphic continuation to all $s \in \mathbb{C}$. When this is combined with the functional equation of theorem 2.3.6 we immediately obtain the meromorphic continuation of the local L-function to the whole complex plane.
□

§2.4 Classical L-functions and root numbers

The founders of the theory of L-functions over $\mathbb{Q}$, Riemann, Dirichlet, Hecke, etc., defined L-functions so that their Euler products were as simple as possible and so that their functional equations were as symmetric as possible. The classical L-functions over $\mathbb{Q}$ are just the Dirichlet L-functions as in (2.2.20) or the Riemann zeta function.

If we are given a Schwartz function $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ as in definition 2.2.2 and an adelic automorphic form $\omega : GL(1, \mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ as in definition 2.1.4, then we have intensively studied Tate's global L-function which was defined by

$$L(s, f, \omega) := \int\limits_{GL(1,\mathbb{A}_\mathbb{Q})} f(x)\, \omega(x)\, |x|_\mathbb{A}^s \, d^\times x$$

for $\Re(s) > 1$. It is natural to ask which choices of f and ω lead to the classical L-functions?

Let $s \in \mathbb{C}$ with $\Re(s) > 1$. If we assume that $f = \prod_v f_v$ is factorizable and $\omega = \prod_v \omega_v$ then

$$L(s, f, \omega) = \prod_{v \le \infty} \int\limits_{\mathbb{Q}_v^\times} f_v(x_v)\, \omega_v(x_v)\, |x_v|_v^s \, d^\times x_v = \prod_{v \le \infty} L_v(s, f_v, \omega_v), \tag{2.4.1}$$

which reduces Tate's global L-function to a product of local L-functions. Recall that a local character $\omega_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ may be ramified or unramified as in definition

2.1.14. Accordingly, we say the prime v is ramified or unramified, respectively. In order to construct the classical L-functions it is necessary to choose f_v (for every $v \le \infty$) so that the local integral (2.4.1) is as simple as possible and the global functional equation of $L(s, f, \omega)$ is as symmetric as possible. It is easy to do this if v is unramified . In this case, we choose f_v (for all $v \le \infty$) so that $\hat{f}_v = f_v$, i.e., f_v is its own Fourier transform. This amounts to the choices:

$$f_v(x_v) = \begin{cases} e^{-\pi x_\infty^2}, & \text{if } v = \infty \text{ is unramified,} \\ 1_{\mathbb{Z}_p}(x_p), & \text{if } v = p < \infty \text{ is unramified.} \end{cases}$$

Define the local L-function, denoted $L_v(s, \omega_v)$, by setting $L_v(s, \omega_v) = L_v(s, f_v, \omega_v)$ for the above choice of f_v. It follows from (2.4.1) and the proof of theorem 2.3.10 that the local L-functions take the form

$$L_v(s, \omega_v) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), & \text{if } v = \infty \text{ is unramified,} \\ \left(1 - \frac{\omega_p(p)}{p^s}\right)^{-1}, & \text{if } v = p < \infty \text{ is unramified.} \end{cases} \tag{2.4.2}$$

It is not as clear, however, how to make the choices when v is ramified. Again, using theorem 2.3.10 as our guide, we will choose f_v so that $L_v(s, \omega_v) := L_v(s, f_v, \omega_v)$ and

$$L_v(s, \omega_v) = \begin{cases} \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right), & \text{if } v = \infty \text{ is ramified,} \\ 1, & \text{if } v = p < \infty \text{ is ramified.} \end{cases} \tag{2.4.3}$$

We may then define the classical completed L-function

$$L^*(s, \omega) = \prod_{v \le \infty} L_v(s, \omega_v) \tag{2.4.4}$$

where $L_v(s, \omega_v)$ is defined in (2.4.2), (2.4.3). Then

$$L^*(s, \omega) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \prod\limits_{p \text{ unramified}} \left(1 - \frac{\omega_p(p)}{p^s}\right)^{-1}, & \text{if } \infty \text{ is unramified,} \\ \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right) \prod\limits_{p \text{ unramified}} \left(1 - \frac{\omega_p(p)}{p^s}\right)^{-1}, & \text{if } \infty \text{ is ramified.} \end{cases}$$

Definition 2.4.5 (Local root number) *Fix $v \le \infty$. Let $L_v(s, \omega_v)$ be given by (2.4.2), (2.4.3), and for a Schwartz function $f_v : \mathbb{Q}_v \to \mathbb{C}$, let $L_v(s, f_v, \omega_v)$ be a non identically vanishing local L-function as in (2.3.4). We define the local root number to be the complex valued function $\epsilon_v(s, \omega_v)$ which satisfies*

$$\frac{L_v(1-s, \hat{f}_v, \bar{\omega}_v)}{L_v(1-s, \bar{\omega}_v)} = \epsilon_v(s, \omega_v)\frac{L_v(s, f_v, \omega_v)}{L_v(s, \omega_v)}. \tag{2.4.6}$$

Note that by theorem 2.3.6, the local root number $\epsilon_v(s, \omega_v)$ is independent of the choice of f_v. Furthermore, by theorem 2.3.10, the local root number $\epsilon_v(s, \omega_v)$ takes the value 1 at unramified primes p. It follows that the infinite product

$$\boxed{\epsilon(s, \omega) := \prod_{v \le \infty} \epsilon_v(s, \omega_v)} \tag{2.4.7}$$

is really just a finite product which can be explicitly computed using theorem 2.3.10.

Proposition 2.4.8 (Global functional equation) *For $\Re(s) > 1$, let $L^*(s,\omega)$ be defined as in (2.4.4). Then $L^*(s,\omega)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with at most simple poles at $s = 0, 1$, and satisfies the functional equation*

$$L^*(s,\omega) = \epsilon(s,\omega)L^*(1-s,\bar{\omega})$$

where $\epsilon(s,\omega)$ is given by (2.4.7). If ω is not the trivial character then $L^*(s,\omega)$ is an entire function of s.

Proof: For each prime $v \le \infty$, define a local L-function $L(s, f_v^\circ, \omega_v)$ as in (2.3.11), (2.3.12), (2.3.13), (2.3.14). For $\Re(s) > 1$ let

$$L(s, f^\circ, \omega) = \prod_{v \le \infty} L(s, f_v^\circ, \omega_v).$$

Then by (2.2.17), we know that $L(s, f^\circ, \omega)$ has a meromorphic continuation to all $s \in \mathbb{C}$, with at most simple poles at $s = 0, 1$, and satisfies the global functional equation

$$L(s, f^\circ, \omega) = L(1-s, \widehat{f^\circ}, \omega). \tag{2.4.9}$$

Now $L(s, f^\circ, \omega)$ is almost the same as $L^*(s,\omega)$. When these functions are defined for $\Re(s) > 1$ as products, then they differ only at the primes p where ω_p is ramified. In fact, for $\Re(s) > 1$, we may write

$$\frac{L(s, f^\circ, \omega)}{L^*(s,\omega)} = \prod_{p \text{ ramified}} \frac{L(s, f_p^\circ, \omega_p)}{L(s,\omega_p)} = \prod_{p \text{ ramified}} L(s, f_p^\circ, \omega_p). \tag{2.4.10}$$

It follows from (2.3.14) that if ω_p is ramified then $L(s, f_p^\circ, \omega_p)$ is just a constant times a power of p^s. Thus the right side of (2.4.10) is holomorphic everywhere and never zero. This shows that $L^*(s,\omega)$ has the same properties as $L(s, f^\circ, \omega)$. In particular it has meromorphic continuation to all complex s with at most simple poles at $s = 0, 1$. The simple poles can only occur when ω is trivial and we are in the situation of the Riemann zeta function.

Further, by the definition of the local root number (2.4.6), (2.4.7), we obtain from (2.4.10) that

$$\frac{L(s, f^\circ, \omega)}{L^*(s,\omega)} = \epsilon(s,\omega)^{-1} \prod_{p \text{ ramified}} \frac{L(1-s, \widehat{f_p^\circ}, \overline{\omega}_p)}{L(1-s, \overline{\omega}_p)}. \tag{2.4.11}$$

The function on the right side of (2.4.10) has meromorphic continuation to $\Re(s) < 0$, and in that region is equal to

$$\epsilon(s,\omega)^{-1} \cdot \frac{L(1-s, \widehat{f^\circ}, \overline{\omega})}{L^*(1-s, \overline{\omega})},$$

from which we deduce that

$$\frac{L(1-s,\widehat{f^\circ},\overline{\omega})}{L^*(1-s,\overline{\omega})}=\epsilon(s,\omega)\frac{L(s,f^\circ,\omega)}{L^*(s,\omega)}. \tag{2.4.12}$$

When (2.4.12) is combined with (2.4.9), proposition 2.4.8 follows. □

Remarks on roots numbers: The global root number $\epsilon(s,\omega)$ is precisely Tate's root number as defined in 2.2.21.

§2.5 Automorphic representations for $GL(1,\mathbb{A}_{\mathbb{Q}})$

We begin with the abstract definition of a representation of a group on a vector space.

Definition 2.5.1 (Representation of a group on a vector space) *Let G be a group and let V be a vector space. A representation of G on V is a homomorphism*

$$\pi: G\to \mathrm{End}(V)=\{\textit{set of all linear maps: } V\to V\}.$$

Here $\pi(g)\,.\,v$ denotes the action of $\pi(g)$ on v and $\pi(g'g'')=\pi(g')\,.\,\pi(g'')$ for all $g',g''\in G$. We call V the space of π and refer to the ordered pair (π,V) as a representation.

Remarks: Very often we shall consider representations (π,V) where V is a space of functions $f: G\to\mathbb{C}$ and the action is given by right translation, i.e.,

$$\pi(g')\,.\,f(g)=f(gg'),\qquad (\forall\, g,g'\in G).$$

In this situation we see that $\pi(g'g'')\,.\,f(g)=\pi(g')\,.\,(\pi(g'')\,.\,f(g))$. Note that a representation defined by a left action on a space of functions would satisfy the reverse identity $\pi(g'g'')=\pi(g'')\,.\,\pi(g')$ for $g',g''\in G$.

If the group G and the vector space V are equipped with topologies, then we shall also require the map $G\times V\to V$, given by $(g,v)\to\pi(g)\,.\,v$, to be continuous.

Definition 2.5.2 (Irreducible representation) *A representation (π,V) as in 2.5.1 is said to be irreducible if $V\neq 0$, and V has no closed π-invariant subspace other than 0 and V.*

Definition 2.5.3 (Intertwining maps and isomorphic representations) *Let*

$$\pi_1: G\to\mathrm{End}(V_1),\qquad \pi_2: G\to\mathrm{End}(V_2),$$

be two representations as in definition 2.5.1. An intertwining map, or intertwining operator, is a linear map $L: V_1\to V_2$ such that

$$L\,.\,\big(\pi_1(g)\,.\,v\big)\;=\;\pi_2(g)\,.\,(L\,.\,v)$$

for all $g\in G$, $v\in V_1$. Here $L\,.\,v$ denotes the action of L on $v\in V_1$. If there is an intertwining operator $L: V_1\to V_2$ which an isomorphism of vector spaces, then the two representations are said to be isomorphic.

Roughly speaking, an automorphic representation of $GL(n)$ (for $n = 1, 2, \dots$) is an irreducible representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$ on a topological vector space V which consists of complex valued functions on $GL(n, \mathbb{A}_{\mathbb{Q}})$ satisfying certain growth, smoothness, and invariance properties. We shall now make the definition precise for the case of $GL(1, \mathbb{A}_{\mathbb{Q}})$.

Definition 2.5.4 (Automorphic representation for $GL(1, \mathbb{A}_{\mathbb{Q}})$) *Fix a unitary Hecke character ω of $\mathbb{A}_{\mathbb{Q}}^{\times}$ as in (2.1.2). Define V_ω to be the one-dimensional vector space (over $\mathbb{C}$) of all automorphic forms with character ω as defined in 2.1.4. We may define a representation*

$$\pi : GL(1, \mathbb{A}_{\mathbb{Q}}) \to \text{End}\,(V_\omega)$$

by requiring that

$$\pi(g) \,.\, \phi(x) := \phi(x \cdot g) = \omega(g)\phi(x)$$

for all $\phi \in V_\omega$ and all $g, x \in GL(1, \mathbb{A}_{\mathbb{Q}})$.

Remarks: It is clear that the Hecke character ω is factorizable, as a function, in the sense described at the beginning of this chapter. But in fact, the representation (π, V_ω) also factors *as a representation.* Once everything is viewed in the right manner, this fact is very close to being a tautology. We use this as an opportunity to introduce the notion of "factorizable" which is appropriate to representations. We first introduce the appropriate product, which is the *restricted tensor product.* In a sense, this product is to tensor products what the restricted direct product used to form the adeles is to Cartesian products.

We first recall the ordinary tensor product of two vector spaces (see [Atiyah-Macdonald, 1969]. Suppose V and W are vector spaces over a field k, with (possibly infinite) bases B and C. Then the tensor product of V and W is a vector space, denoted $V \otimes W$, generated by a basis consisting of the symbols $v \otimes w$, where $v \in B$ and $w \in C$. It may also be realized as the vector space spanned by the set of *all* symbols $v \otimes w$ with $v \in V$ and $w \in W$, such that the only relations are those coming from relations in V, relations in W, or relations of the form

$$\alpha \cdot (v \otimes w) = (\alpha \cdot v) \otimes w = v \otimes (\alpha \cdot w), \qquad (\text{for all } \alpha \in k,\ v \in V,\ w \in W),$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \qquad (\text{for all } v_1, v_2 \in V,\ w \in W),$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \qquad (\text{for all } v \in V,\ w_1, w_2 \in W).$$

An element of the form $v \otimes w$ is called a "pure tensor." Because the pure tensors span, it is often enough to prove theorems only for pure tensors.

This construction generalizes in a straightforward way to threefold and n-fold tensor products, and even to infinite tensor products. The tensor products of interest to us will be infinite and indexed by the set of primes. For each prime v let V_v be a vector space. For these, we write $\otimes_v V_v$ for the tensor product and $\otimes_v \xi_v$ for a pure tensor, where $\xi_v \in V_v$ for each v.

To define a *restricted* tensor product, one must fix a nonzero vector ξ_v° in V_v for all but finitely many v. Then the restricted tensor product of $\{V_v\}$ relative to $\{\xi_v^\circ\}$ is the subspace of $\otimes_v V_v$ spanned by the set of pure tensors which are of the form

$$\xi = \otimes_v \xi_v \qquad \xi_v = \xi_v^\circ \text{ for almost all } v.$$

This vector space is denoted $\otimes'_v V_v$.

Definition 2.5.5 (Factorizability for representations of $\mathbb{A}_{\mathbb{Q}}^{\times}$) *Let V be a vector space. A representation (π, V) of $\mathbb{A}_{\mathbb{Q}}^{\times}$ is factorizable if there exists:*

(1) *For each v, a representation π_v of $\mathbb{Q}_v^{\times}$, on a vector space V_v.*

(2) *For almost all v, a distinguished vector $\xi_v^{\circ} \in V_v$.*

(3) *An isomorphism of vector spaces $\ell : \otimes_v' V_v \to V$,*

such that

$$\ell\Big(\otimes_v \left(\pi_v(g_v) \, . \, \xi_v\right)\Big) = \pi(g) \, . \, \ell(\otimes_v \xi_v), \tag{2.5.6}$$

for all $g = \{g_\infty, g_2, g_3, \dots\} \in \mathbb{A}_{\mathbb{Q}}^{\times}$ and all $\otimes_v \xi_v$ such that $\xi_v = \xi_v^{\circ}$ for almost all v.

Remarks: The linear map ℓ in definition 2.5.5 gives an isomorphism of $\otimes_v' V_v$ and V. Note that (2.5.6) is precisely what is required (see definition 2.5.3) to make the representation $(\otimes_v \pi_v, \ \otimes_v' V_v)$ isomorphic to the representation (π, V).

Proposition 2.5.7 (All automorphic representations for $GL(1, \mathbb{A}_{\mathbb{Q}})$ are factorizable) *The representation (π, V_ω) as defined in 2.5.4 is factorizable.*

Proof: For each v, we take an abstract one-dimensional complex vector space, spanned by a vector, which we denote ξ_v°. Thus V_v is simply $\{c\xi_v^{\circ} \mid c \in \mathbb{C}\}$ for each v. We define an action of $\mathbb{Q}_v^{\times}$ on V_v by

$$\pi_v(g) \, . \, (c\xi_v^{\circ}) = c \, \omega_v(g) \, \xi_v^{\circ}, \qquad (g \in \mathbb{Q}_v^{\times}).$$

Now, we need to define a linear map ℓ from $\otimes_v' V_v$ to V_ω. By linearity, it is enough to define it on the element $\otimes_v \xi_v^{\circ}$, which spans $\otimes_v' V_v$. Recall that V_ω is a one-dimensional vector space spanned by the function ω. The definition of ℓ is simply

$$\ell\big(\otimes_v \xi_v^{\circ}\big) = \omega.$$

One then easily verifies that

$$\ell\Big(\otimes_v \left(\pi_v(g_v) \, . \, \xi_v\right)\Big) = \pi(g) \, . \, \ell\big(\otimes_v \xi_v\big).$$

(Both sides are equal to $\omega(g)\omega$, i.e, the function in V_ω which takes the value $\omega(g)\omega(x)$ at the point x.) $\square$

§2.6 Hecke operators for $GL(1, \mathbb{A}_{\mathbb{Q}})$

The theory of Hecke operators for automorphic forms on $GL(2, \mathbb{R})$ is well known (see [Goldfeld, 2006]). Very roughly, Hecke operators act on automorphic forms and transform them to other automorphic forms. If an automorphic form is an eigenfunction of all the Hecke operators then it is called a Hecke eigenform. Hecke proved the remarkable theorem that the L-function associated to a Hecke eigenform has an Euler product. We shall now show that a similar result can be established for automorphic forms for $GL(1, \mathbb{A}_{\mathbb{Q}})$.

Definition 2.6.1 (Hecke operator for $GL(1,\mathbb{A}_\mathbb{Q})$) *Let ϕ be an automorphic form for $GL(1,\mathbb{A}_\mathbb{Q})$ with unitary Hecke character ω as in definition 2.1.4. For every idele $n = \{n_\infty, n_2, \dots\} \in \mathbb{A}_\mathbb{Q}^\times$ we define the Hecke operator T_n which acts on ϕ as follows*

$$T_n\phi(g) = \phi(ng)$$

for all ideles $g \in GL(1,\mathbb{A}_\mathbb{Q})$.

By definition 2.1.4 (2), we see that

$$T_n\phi(g) = \omega(n)\phi(g)$$

so that every automorphic form is a Hecke eigenform. Actually, definition 2.6.1 is not very interesting because the space of automorphic forms on $GL(1,\mathbb{A}_\mathbb{Q})$ with fixed character ω is just a one-dimensional space. Hecke's theorem is trivial in this situation because we have already shown that the only possible L-functions we can obtain for $GL(1,\mathbb{A}_\mathbb{Q})$ are Dirichlet L-functions which do have Euler products. We state Hecke's theorem for completeness.

Theorem 2.6.2 *Let ϕ be an automorphic form for $GL(1,\mathbb{A}_\mathbb{Q})$ with unitary Hecke character ω as in definition 2.1.4. If ϕ is an eigenfunction of all the Hecke operators T_n as defined in 2.6.1 then the L-function associated to ϕ has an Euler product.*

§2.7 The Rankin-Selberg method

The purpose of section 2.7 is to show that the classical Rankin-Selberg method (see [Goldfeld, 2006]) has an analogue on $GL(1,\mathbb{A}_\mathbb{Q})$. This very brief section is for cognoscenti who already have familiarity with the classical Rankin-Selberg method.

Let $s \in \mathbb{C}$ with $\Re(s) > 0$. Let $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ be an adelic Schwartz function as in definition 1.7.4. The Tate series associated to f is defined to be

$$T(x,s,f) := \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\cdot|\alpha x|_\mathbb{A}^s = \sum_{\alpha\in\mathbb{Q}^\times} f(\alpha x)\cdot|x|_\mathbb{A}^s$$

for all $x \in \mathbb{A}_\mathbb{Q}^\times$. Because we are averaging over the multiplicative group $\mathbb{Q}^\times$ it is easy to see that

$$T(\gamma x,s,f) = T(x,s,f)$$

for all $\gamma \in \mathbb{Q}^\times$ and $x \in \mathbb{A}_\mathbb{Q}^\times$. If ϕ_1, ϕ_2 are automorphic forms for $GL(1,\mathbb{A}_\mathbb{Q})$ with characters ω_1, ω_2, respectively, as in definition 2.1.4, then it is clear that $\phi_1\cdot\phi_2$ is again an automorphic form with character $\omega_1\cdot\omega_2$. The classical Rankin-Selberg unfolding computation takes the following form on $GL(1,\mathbb{A}_\mathbb{Q})$:

$$\int\limits_{\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times} \phi_1(x)\phi_2(x)T(x,s,f)\,d^\times x = \int\limits_{\mathbb{A}_\mathbb{Q}^\times} \phi_1(x)\phi_2(x)f(x)|x|_\mathbb{A}^s\,d^\times x \tag{2.7.1}$$

where the right side of (2.7.1) is the completed L-function (product of local L-functions) associated to the automorphic form $\phi_1\cdot\phi_2$.

§2.8 The p-adic Mellin transform

Definition 2.8.1 (p-adic Mellin transform) *Let $f : \mathbb{Q}_p^\times \to \mathbb{C}$ be a locally constant compactly supported function. Assume that $f(y) = h(|y|_p)$ for some other function $h : \{p^\ell \mid \ell \in \mathbb{Z}\} \to \mathbb{C}$. For $s \in \mathbb{C}$, we define the p-adic Mellin transform*

$$\widetilde{f}(s) := \int_{\mathbb{Q}_p^\times} f(u)\,|u|_p^s\,d^\times u.$$

Proposition 2.8.2 (p-adic Mellin inversion) *Let $f : \mathbb{Q}_p^\times \to \mathbb{C}$ be a locally constant compactly supported function. Assume that $f(y) = h(|y|_p)$ for some other function $h : \{p^\ell \mid \ell \in \mathbb{Z}\} \to \mathbb{C}$. Let $\widetilde{f}$ be the Mellin transform as in definition 2.8.1. Then*

$$f(y) = \frac{\log p}{2\pi} \int_0^{\frac{2\pi}{\log p}} \widetilde{f}(it)\,|y|_p^{-it}\,dt.$$

Proof: It follows from definition 2.8.1 that for $|y|_p = p^\ell$, we have

$$\begin{aligned}
\frac{\log p}{2\pi} \int_0^{\frac{2\pi}{\log p}} \tilde{f}(it)\,|y|_p^{-it}\,dt &= \frac{\log p}{2\pi} \int_0^{\frac{2\pi}{\log p}} \left[\int_{\mathbb{Q}_p^\times} f(u)\,|u|_p^{it}\,d^\times u\right] p^{-i\ell t}\,dt \\
&= \sum_{m\in\mathbb{Z}} h\left(p^m\right) \int_{p^m\mathbb{Z}_p^\times} \left[\frac{\log p}{2\pi} \int_0^{\frac{2\pi}{\log p}} p^{i(m-\ell)t}\,dt\right] d^\times u \\
&= h\left(p^\ell\right). \quad \square
\end{aligned}$$

The above Mellin transform and its inverse can be generalized to compactly supported locally constant functions $f : \mathbb{Q}_p^\times \to \mathbb{C}$ where $f(y)$ is not necessarily a function of $|y|_p$. It is convenient to make the following definition.

Definition 2.8.3 (Conductor of a locally constant compactly supported function) *Let $f : \mathbb{Q}_p \to \mathbb{C}$ be a locally constant compactly supported function. Then f must be identically equal to zero on $p^m\mathbb{Z}_p^\times$ for all but a finite number of integers m. Let $\mathcal{M}$ denote the finite set of such integers m. For each $m \in \mathcal{M}$, there exists an integer N_m such that the function $f(p^m y)$ must be constant on the cosets of $1 + p^N\mathbb{Z}_p$ in $\mathbb{Z}_p^\times$. Choose N to be the largest of the values of all such N_m with $m \in \mathcal{M}$. Then p^N is defined to be the conductor of f.*

Definition 2.8.4 (Normalized unitary character of $\mathbb{Q}_p^\times$) *A continuous function $\psi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ which satisfies*

- $\psi(yy') = \psi(y)\psi(y'), \quad \left(\forall y, y' \in \mathbb{Q}_P^\times\right);$
- $|\psi(y)|_\mathbb{C} = 1, \quad \left(\forall y \in \mathbb{Q}_p^\times\right),$
- $\psi(p) = 1,$

is called a normalized unitary multiplicative character of $\mathbb{Q}_p^\times$. *Let* N *be the smallest integer* $k \geq 0$ *such that* $1 + p^k\mathbb{Z}_p$ *is contained in the kernel of* ψ. *Then* ψ *is said to have conductor* p^N. *We also call* ψ *a character* $(\text{mod } p^N)$.

The number of $\psi \,(\text{mod } p^N)$ is $\varphi(p^N)$, where φ is Euler's function. This follows from (2.1.15), along with the corresponding fact about classical Dirichlet characters. With these preliminaries in place, we may now present the more general Mellin transform.

Definition 2.8.5 (General p-adic Mellin transform) *Let* $f : \mathbb{Z}_p^\times \to \mathbb{C}$ *be a locally constant compactly supported function. For* $s \in \mathbb{C}$, *and* $\psi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, *a normalized unitary character as in 2.8.4, we define the Mellin transform*

$$\widetilde{f}(s, \psi) := \int\limits_{\mathbb{Q}_p^\times} f(u)\,\psi(u)\,|u|_p^s\, d^\times u.$$

Proposition 2.8.6 (General p-adic Mellin inversion) *Let* $f : \mathbb{Z}_p^\times \to \mathbb{C}$ *be a locally constant compactly supported function of conductor* p^N *as in 2.8.3. Let* $\widetilde{f}$ *be the Mellin transform as in definition 2.8.5. Then we have*

$$f(y) = \frac{1}{\varphi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \frac{\log p}{2\pi} \int\limits_0^{\frac{2\pi}{\log p}} \widetilde{f}(it, \psi)\,\psi(y)^{-1}|y|_p^{-it}\, dt.$$

Proof: It follows from definition 2.8.5 that if $|y|_p = p^\ell$, then

$$\begin{aligned}
&\frac{1}{\varphi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \frac{\log p}{2\pi} \int\limits_0^{\frac{2\pi}{\log p}} \widetilde{f}(it, \psi)\,\psi(y)^{-1}|y|_p^{-it}\, dt \\
&= \frac{1}{\varphi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \frac{\log p}{2\pi} \int\limits_0^{\frac{2\pi}{\log p}} \left[\, \int\limits_{\mathbb{Q}_p^\times} f(u)\,\psi(u)\,|u|_p^{it}\, d^\times u \right] \psi(y)^{-1}|y|_p^{-it}\, dt \\
&= \frac{1}{\phi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \int\limits_{\mathbb{Q}_p^\times} f(u)\,\psi(u)\;\psi(y)^{-1} \left[\frac{\log p}{2\pi} \int\limits_0^{\frac{2\pi}{\log p}} |u/y|_p^{-it}\, dt \right] d^\times u \\
&= \frac{1}{\varphi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \int\limits_{p^\ell \mathbb{Z}_p^\times} f(u)\,\psi(u)\;\psi(y)^{-1}\, d^\times u \\
&= f(y).
\end{aligned}$$

The last step in the above argument follows from the orthogonality relation

$$\frac{1}{\varphi(p^N)} \sum_{\psi \,(\text{mod } p^N)} \psi(u)\psi(y)^{-1} = \begin{cases} 1, & \text{if } u - y \in p^N\mathbb{Z}_p, \\ 0, & \text{otherwise,} \end{cases}$$

and the fact that $N \geq \ell$. $\square$

Exercises for Chapter 2

2.1 Let $t \in \mathbb{R}$ be nonzero, and consider the automorphic form $\phi : GL(1,\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ defined by

$$\phi(g) = |g|_{\mathbb{A}}^{it}.$$

Prove that ϕ is continuous, but show that its kernel is not an open subgroup. Deduce that ϕ cannot be the idelic lift of a Dirichlet character.

2.2 From Theorem 2.19, we see that any automorphic form for $GL(1,\mathbb{A}_\mathbb{Q})$ can be written as $\phi(g) = c\,\chi_{\text{idelic}}(g)\,|g|_{\mathbb{A}}^{it}$ for some Dirichlet character χ and some constants c and t. Find an explicit formula for t in terms of "nice" values of ϕ.

2.3 Let G be a topological group (e.g., $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{A}_\mathbb{Q}$) and let $\psi : G \to \mathbb{C}^\times$ be a homomorphism (not necessarily continuous).

(a) Show that ψ is continuous if and only if it is continuous at the identity of G.

(b) If the image of ψ is a finite set, show that ψ is continuous if and only if $\ker(\psi)$ is an open subgroup. (For example, $G = \mathbb{A}_\mathbb{Q}$ and $\psi = \chi_{\text{idelic}}$ for some Dirichlet character χ.)

2.4 Suppose $\omega : \mathbb{Q}_v \to \mathbb{C}^\times$ is a character such that $\omega(-1) = -1$, and let f be an even Schwartz function (i.e., $f(-x) = f(x)$ for all $x \in \mathbb{Q}_v$). Prove that the local L-function satisfies $L(s,f,\omega) = 0$.

2.5 Let $v \le \infty$ be a prime and $\omega_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ a ramified local character. Recall from (2.4.3) that we defined the local factor of the classical completed L-function to be

$$L_v(s,\omega_v) = \begin{cases} \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right), & \text{if } v = \infty, \\ 1 & \text{if } v = p < \infty. \end{cases}$$

Find a Schwarz function $f_v^\circ : \mathbb{Q}_v^\times \to \mathbb{C}$ such that

$$L_v(s, f_v^\circ, \omega_v) = L_v(s,\omega_v).$$

Hint: *Do the cases $v = \infty$ and $v = p < \infty$ separately.*

2.6 For $0 < \Re(s) < 1$ and ω_v any local character, show that

$$\epsilon_v(s,\omega_v) = \frac{L_v(s,\omega_v)}{L_v(1-s,\omega_v)}\rho_{\omega_v}(s)^{-1}.$$

Deduce that $\epsilon_v(s,\omega_v) \equiv 1$ if either $v = \infty$ or $v = p < \infty$ and ω_p is unramified. If $v = p < \infty$ and ω_p is ramified with conductor p^f, then

$$\epsilon_v(s,\omega_v) = p^{-f(s-1)}\omega(p)^f \left(\sum_{\substack{a=1 \\ (a,q)=1}}^{p^f} \omega(a) e^{-2\pi i a/p^f} \right)^{-1}.$$

2.7 For any character $\omega : \mathbb{Q}_v \to \mathbb{C}^\times$ and any complex number s with $0 < \Re(s) < 1$, show that

$$\overline{\rho_\omega(s)} = \omega(-1)\rho_{\overline{\omega}}(\overline{s}).$$

Conclude that

$$|\rho_\omega(1/2)| = 1.$$

When $v = p < \infty$ and ω is ramified with conductor p^f, deduce the following classical fact about Gauss sums:

$$\left| \sum_{\substack{a=1 \\ (a,p)=1}}^{p^f} \omega(a) e^{-2\pi i a/p^f} \right| = \sqrt{p^f}.$$

2.8 Let $\chi \pmod{p^f}$ be a nontrivial Dirichlet character for some prime p and some integer $f \geq 1$.

(a) Show that $\sum\limits_{\substack{j=1 \\ (j,p)=1}}^{p^f} \chi(j) = 0.$

Hint: *Make the change of variables $j \mapsto aj$ for some $a \in \mathbb{Z}$ with $\chi(a) \neq 0$.*

(b) Show that $\int\limits_{\mathbb{Z}_p^\times} \chi_{\text{idelic}}(x_p) d^*x_p = \sum\limits_{\substack{j=1 \\ (j,p)=1}}^{p^f} \bar{\chi}(j) = 0.$

(c) Deduce equation (2.2.19).

2.9 Let G be a group, V a complex vector space, and $\pi : G \to \mathrm{Aut}(V)$ an irreducible representation. Prove that if G is Abelian, then $\dim(V)1$. **Hint:** *For a fixed $g_0 \in G$, consider how G acts on the eigenspaces of $\pi(g_0)$.*

2.10 In this exercise we will show that any continuous unitary character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ is of the form

$$\omega(g) = |g|_\infty^{it}, \qquad \text{or} \qquad \omega(g) = |g|_\infty^{it} \cdot \mathrm{sign}(g),$$

for some fixed constant $t \in \mathbb{R}$ as in (2.1.11). In order to accomplish this, we will assume the following fact from the theory of covering spaces in algebraic topology: for any continuous map $f : \mathbb{R}^+ \to \mathbb{C}^\times$ with values in the unit circle, there exists a *continuous* map $\lambda : \mathbb{R}^+ \to \mathbb{R}$ such that $\lambda(1) = 0$ and $f(g) = e^{i\lambda(g)}$. Here $\mathbb{R}^+$ is the multiplicative group of positive real numbers. The existence of such a map λ is reasonably obvious, but the nontrivial part is that we may take λ to be continuous.

For the remainder of the problem, let ω be any continuous unitary character as above, and let $f = \omega|_{\mathbb{R}^+}$ be the restriction of ω to the subgroup of positive reals.

(a) Define a map $\Lambda : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by the formula

$$\Lambda(g, h) = \lambda(gh) - \lambda(g) - \lambda(h).$$

Show that $e^{i\Lambda(g,h)} = 1$ for all $g, h \in \mathbb{R}^+$. Conclude that λ is a group homomorphism. **Hint:** *You will need to use the continuity of λ for the final conclusion.*

(b) Show that there exists $t \in \mathbb{R}$ such that $\lambda(e^r) = rt$ for all real numbers r. Deduce that $f(g) = g^{it}$ for all $g \in \mathbb{R}^+$.

(c) Show that

$$\omega(g) = \begin{cases} |g|_\infty^{it}, & \text{if } \omega(-1) = 1, \\ |g|_\infty^{it} \cdot \operatorname{sign}(g), & \text{if } \omega(-1) = -1. \end{cases}$$

2.11 Let p be a prime number. Recall that a character of $\mathbb{Q}_p^\times$ is a continuous homomorphism $\omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. If ω is a character of $\mathbb{Q}_p^\times$ with image in the positive real numbers, show that ω is unramified.

2.12 Fix a prime number p. Prove that an unramified continuous unitary character $\omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ can be written as $\omega(x) = |x|_p^{it}$ for some real number t. Show also that t is uniquely determined up to addition of an integer multiple of $2\pi/\log p$.

2.13 In this exercise, we give two examples of representations of the topological group $\mathbb{R}$, the additive group of real numbers endowed with its usual topology. One example is a continuous representation, while the other is not.

(a) Let $\mathbb{R}^2$ be the standard 2-dimensional vector space over the field $\mathbb{R}$ endowed with its usual topology. Show that the map $\pi : \mathbb{R} \to GL(2, \mathbb{R})$ defined by $\pi(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ determines a continuous representation of $\mathbb{R}$ on $\mathbb{R}^2$. That is, verify that $\pi(uu').v = \pi(u).(\pi(u').v)$ for all $u, u' \in \mathbb{R}$ and that the associated map $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $(u, v) \mapsto \pi(u).v$ is continuous.

(b) With the setup as in part (a), prove that there exists a homomorphism $f : \mathbb{R} \to \mathbb{R}$ that is *not continuous*. Conclude that the map $\pi : \mathbb{R} \to GL(1, \mathbb{R})$ given by $\pi(a).v = e^{f(a)}v$ is a group representation, but that the associated map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(a, v) \mapsto e^{f(a)}v$ is not continuous.