Completions and completeness of normed algebras of differentiable functions

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Abstract

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This is joint work with Professor H. Garth Dales (Leeds).

We investigate the completeness and completions of the normed algebras $D^{(1)}(X)$ of continuously complex-differentiable functions on perfect compact plane sets X (introduced by Dales and Davie in 1973).

We solve some problems raised in an earlier paper of Bland and Feinstein (2005) by constructing a variety of compact plane sets X with dense interior such that $D^{(1)}(X)$ is not complete.

We also show that the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

The algebra $D^{(1)}(X)$

Throughout, by **compact plane set** we shall mean an **infinite**, compact subset of $\mathbb{C}$.

Let X be a compact plane set. We denote the set of all continuous, complex-valued functions on X by $C(X)$.

For $f \in C(X)$, we denote the uniform norm of f on a non-empty subset E of X by $|f|_E$.

Definition

Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We then call $f'(a)$ the (**complex**) **derivative** of f at a.

Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

For $f \in D^{(1)}(X)$, set

$$\|f\| = |f|_X + |f'|_X \,.$$

Then $(D^{(1)}(X), \|\cdot\|)$ is easily seen to be a normed algebra.

The normed algebra $D^{(1)}(X)$ is often incomplete, even for fairly nice X.

Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

The character space of $D^{(1)}(X)$

We now investigate the character space of $D^{(1)}(X)$.

The proof of the next theorem is based on the method used by Jarosz (1997) to determine the character space of $\mathrm{Lip}_{Hol}(X, \alpha)$.

Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \|\cdot\|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.
In particular, every character on $(D^{(1)}(X), \|\cdot\|)$ is continuous.

Proof.

Let ϕ be a character on A, and set $w = \phi(Z)$, where Z is the coordinate functional (restricted to X in this setting).

Then $\phi(Z - w1) = 0$, and so $Z - w1$ is not invertible in A.

Since every rational function with poles off X is in A, it follows that $w \in X$.

We show that ϕ is the point evaluation character at w, ε_w.

To see this, it is sufficient to show that $\ker(\varepsilon_w) \subseteq \ker(\phi)$.

Take $f \in A$ with $f(w) = 0$.

Since

$$\lim_{z\to w,\, z\in X} \frac{f(z)}{z-w} = f'(w)\,,$$

it follows that there is a positive constant C such that, for all $z \in X$, $|f(z)| \leq C|z-w|$.

It is now easy to see that $f^3 = (Z - w1)g$ for a (unique) function $g \in D^{(1)}(X)$ (with $g(w) = g'(w) = 0$).

This gives

$$(\phi(f))^3 = \phi(f^3) = \phi(Z - w1)\phi(g) = 0\,,$$

and so $\phi(f) = 0$.

The result follows. $\square$

Rectifiable paths

We shall assume that the reader is familiar with the elementary results and definitions concerning rectifiable paths including integration of continuous, complex-valued functions along rectifiable paths.

Definition

A **path** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a < b$; γ is a path **from** $\gamma(a)$ **to** $\gamma(b)$ with **endpoints** $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

A **subpath** of γ is then any path obtained by restricting γ to a non-degenerate closed sub-interval of $[a, b]$.

Given $X \subseteq \mathbb{C}$, a **path in** X is a path in $\mathbb{C}$ whose image is a subset of X, and a **Jordan arc in** X is an injective path in X.

The length of a rectifiable path γ will be denoted by $|\gamma|$.

A path in $\mathbb{C}$ is **admissible** if it is rectifiable and has no constant subpaths.

Definition

Let X be a compact plane set.

We say that X is **rectifiably connected** if, for all z and w in X, there is a rectifiable path γ from z to w in X.

Suppose now that X is rectifiably connected.

For z and w in X, we denote the geodesic distance between z and w by $\delta(z, w)$.

We say that such a compact plane set X is **geodesically bounded** if X is is bounded with respect to the metric δ.

We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

Definition

Let X be a compact plane set.

Let $z \in X$. The set X is **regular at** z if there is a constant $k_z > 0$ such that, for every $w \in X$, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k_z|z - w|$.

The set X is **pointwise regular** if X is regular at every point $z \in X$.

The set X is **uniformly regular** if there is one constant $k > 0$ such that, for all z and w in X, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k|z - w|$.

Dales and Davie showed that $D^{(1)}(X)$ is complete whenever X is a finite union of uniformly regular, compact plane sets.

Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

Related spaces

We now recall some related spaces which were discussed in the paper of Bland and Feinstein.

For an open subset U of $\mathbb{C}$, $O(U)$ is the algebra of analytic functions on U.

Now let X be a compact plane set and, throughout, set $U = \operatorname{int} X$.

Then

$$A(X) = \{f \in C(X) : f|_U \in O(U)\}.$$

Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

In this setting, we set

$$\|f\| = |f|_X + |f'|_U \quad (f \in A^{(1)}(X)).$$

Then $(A^{(1)}(X), \|\cdot\|)$ is a Banach function algebra on X, and there is an obvious isometric inclusion of $D^{(1)}(X)$ in $A^{(1)}(X)$.

The completion of $D^{(1)}(X)$ is then its closure in $(A^{(1)}(X), \|\cdot\|)$.

The above is only helpful for compact plane sets X such that U is dense in X.

This is too restrictive for our purposes, and instead we shall mostly work with the larger class of compact plane sets X for which the union of the images of all admissible rectifiable paths in X is dense in X.

We begin by defining a new term, ‘effective’, which is a modification of the term ‘useful’ introduced by Bland and Feinstein.

Definition

Let X be a compact plane set, and let $\mathcal{F}$ be a family of paths in X.

Then $\mathcal{F}$ is **effective** if each subpath of a path in $\mathcal{F}$ belongs to $\mathcal{F}$, if each path in $\mathcal{F}$ is rectifiable and non-constant, and the union of the images of the paths in $\mathcal{F}$ is dense in X.

Note that, if $\mathcal{F}$ is effective, then every path in $\mathcal{F}$ is admissible.

We shall often take $\mathcal{F}$ to be the set of all admissible paths in X.

In this case, $\mathcal{F}$ is effective if and only if the union of the images of all admissible paths in X is dense in X.

The next few definitions and results are essentially as in the paper of Bland and Feinstein, although some proofs require a little more work in the setting of effective families.

Recall that the endpoints of a path γ are denoted by γ^- and γ^+.

Definition

Let $\mathcal{F}$ be a family of rectifiable paths in a compact plane set X.

For $f \in C(X)$, we say that $g \in C(X)$ is an $\mathcal{F}$**-derivative** of f if, for all $\gamma \in \mathcal{F}$, we have

$$\int_\gamma g(z)\,\mathrm{d}z = f(\gamma^+) - f(\gamma^-).$$

We define

$$\mathcal{D}^1_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

If $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ by f'. Moreover, this agrees with the usual derivative for functions in $D^{(1)}(X)$.

Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

For $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, set $\|f\| = |f|_X + |f'|_X$.

Theorem

Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

Then $(\mathcal{D}^1_{\mathcal{F}}(X), \|.\|)$ is a Banach function algebra containing $D^{(1)}(X)$ isometrically as a subalgebra.

The completion of $D^{(1)}(X)$

In this section we discuss the completion of $D^{(1)}(X)$, which we denote by $\widetilde{D}^{(1)}(X)$.

An example of Dales shows that $\widetilde{D}^{(1)}(X)$ need not be semisimple.

However, from above, we now know many settings where $\widetilde{D}^{(1)}(X)$ is a Banach function algebra.

If X is a compact plane set with dense interior, then we know that $\widetilde{D}^{(1)}(X)$ is simply the closure of $D^{(1)}(X)$ in the Banach function algebra $A^{(1)}(X)$

Our next theorem deals with the situation when the union of the images of all admissible rectifiable paths in X is dense in X.

Theorem

Let X be a compact plane set such that the union of the images of all admissible rectifiable paths in X is dense in X.
Then $\widetilde{D}^{(1)}(X)$ is semisimple.

Proof Let $\mathcal{F}$ be the set of all admissible paths in X. Then $\mathcal{F}$ is effective.
We know that $\mathcal{D}^1_{\mathcal{F}}(X)$ is a Banach function algebra, and that we can regard $\widetilde{D}^{(1)}(X)$ as the closure of $D^{(1)}(X)$ in $\mathcal{D}^1_{\mathcal{F}}(X)$.
Thus $\widetilde{D}^{(1)}(X)$ is semisimple. $\square$

For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.
In general there are easy examples where $\mathcal{F}$ is effective but $\widetilde{D}^{(1)}(X) \neq \mathcal{D}^1_{\mathcal{F}}(X)$.

Polynomial approximation

The next result is essentially as in Bland and Feinstein (2005).

Theorem

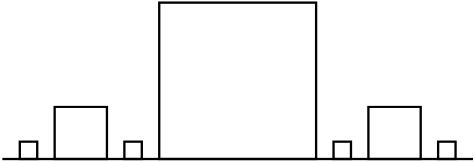
Let X be a polynomially convex, perfect, compact plane set which is geodesically bounded, and let $\mathcal{F}$ be the set of all admissible rectifiable paths in X. Then the (analytic) polynomials are dense in $\mathcal{D}^1_{\mathcal{F}}(X)$, and so $\widetilde{D}^{(1)}(X) = \mathcal{D}^1_{\mathcal{F}}(X)$.

However, when X has dense interior, the corresponding statement for $A^{(1)}(X)$ is false.

Theorem

There exists a uniformly regular, polynomially convex, compact plane set such that X is geodesically bounded and $\operatorname{int} X$ is dense in X and yet $D^{(1)}(X) = \widetilde{D}^{(1)}(X) = \mathcal{D}^1_{\mathcal{F}}(X) \neq A^{(1)}(X)$.
In particular, the polynomials are not dense in $A^{(1)}(X)$.

An example, based on the Cantor middle thirds set, is illustrated in the following diagram.



The Cantor function of x, regarded as a function of $z = x + \mathrm{i}y$, is then in $A^{(1)}(X) \setminus \mathcal{D}^1_{\mathcal{F}}(X)$.

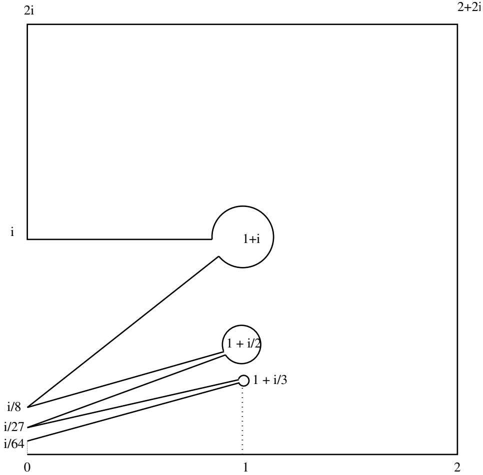
We now give diagrams illustrating two examples to show that $D^{(1)}(X)$ need not be complete when X is polynomially convex, geodesically bounded and has dense interior.

These examples fully solve Problem 6.2 raised in the paper of Bland and Feinstein

Example

There exists a polynomially convex, geodesically bounded compact plane set X such that X has dense interior, but $D^{(1)}(X)$ is incomplete.

Here is a diagram of such an example.



Not to scale

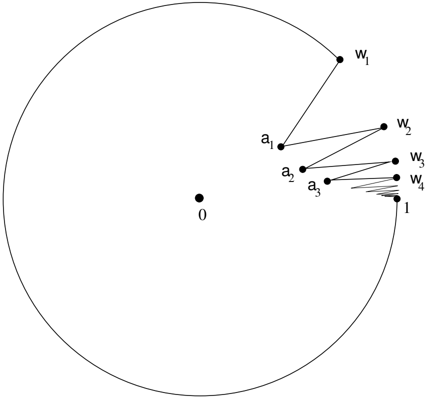
Recall that a compact plane set is **radially self-absorbing** if, for all $r > 1$, $X \subseteq \mathrm{int}\,(rX)$.

Radially self-absorbing, compact plane sets are always polynomially convex and geodesically bounded, and have dense interior.

Example

There exists a radially self-absorbing, compact plane set X such that $D^{(1)}(X)$ is incomplete.

Here is a diagram of such an example.



Here, $w_n = \mathrm{e}^{\mathrm{i}\alpha_n}$ and $a_n = (1 - 4r_n)\mathrm{e}^{\mathrm{i}\beta_n}$, where $\alpha_n = \dfrac{\pi}{4n^2}$, $r_n = \dfrac{1}{8\sqrt{n}}$ and $\beta_n = (\alpha_n + \alpha_{n+1})/2$.

We do not know of an example of a connected, compact plane set X such that X is not pointwise regular, and yet $D^{(1)}(X)$ is complete.

In a result which covers both of the two preceding examples, we have recently managed to eliminate all connected, compact plane sets which fail pointwise regularity at some point z_0 in a similar way due to a sequence of 'dents' approaching z_0.

We conclude with a result which eliminates another class of potential examples, including all rectifiable Jordan arcs.

Theorem

Let X be a polynomially convex, geodesically bounded, compact plane set with empty interior.

If X is not pointwise regular, then $D^{(1)}(X)$ is incomplete.