Completions and completeness of normed algebras of differentiable functions

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Abstract

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We solve some problems raised in an earlier paper of Bland and Feinstein (2005) by constructing a variety of compact plane sets X with dense interior such that $D^{(1)}(X)$ is not complete.

We also show that the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

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Definition

Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\, z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We then call $f'(a)$ the (**complex**) **derivative** of f at a.

Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

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Then $(D^{(1)}(X), \|\cdot\|)$ is easily seen to be a normed algebra.

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Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

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Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \|\cdot\|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

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Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \|\cdot\|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.
In particular, every character on $(D^{(1)}(X), \|\cdot\|)$ is continuous.

Proof.

Let ϕ be a character on A, and set $w = \phi(Z)$, where Z is the coordinate functional (restricted to X in this setting).

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Take $f \in A$ with $f(w) = 0$.

Since

$$\lim_{z \to w,\, z \in X} \frac{f(z)}{z - w} = f'(w)\,,$$

it follows that there is a positive constant C such that, for all $z \in X$, $|f(z)| \leq C|z - w|$.

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It is now easy to see that $f^3 = (Z - w1)g$ for a (unique) function $g \in D^{(1)}(X)$ (with $g(w) = g'(w) = 0$).

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The result follows. $\square$

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A **path** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a < b$; γ is a path **from** $\gamma(a)$ **to** $\gamma(b)$ with **endpoints** $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

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A path in $\mathbb{C}$ is **admissible** if it is rectifiable and has no constant subpaths.

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We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

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Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

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Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

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This is too restrictive for our purposes, and instead we shall mostly work with the larger class of compact plane sets X for which the union of the images of all admissible rectifiable paths in X is dense in X.

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We shall often take $\mathcal{F}$ to be the set of all admissible paths in X.

In this case, $\mathcal{F}$ is effective if and only if the union of the images of all admissible paths in X is dense in X.

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For $f \in C(X)$, we say that $g \in C(X)$ is an $\mathcal{F}$**-derivative** of f if, for all $\gamma \in \mathcal{F}$, we have

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We define

$$\mathcal{D}^1_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

If $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ by f'. Moreover, this agrees with the usual derivative for functions in $D^{(1)}(X)$.

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Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

For $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, set $\|f\| = |f|_X + |f'|_X$.

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Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

Then $(\mathcal{D}^1_{\mathcal{F}}(X), \|.\|)$ is a Banach function algebra containing $D^{(1)}(X)$ isometrically as a subalgebra.

The completion of $D^{(1)}(X)$

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.

In general there are easy examples where $\mathcal{F}$ is effective but $\widetilde{D}^{(1)}(X) \neq \mathcal{D}^1_{\mathcal{F}}(X)$.

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In particular, the polynomials are not dense in $A^{(1)}(X)$.

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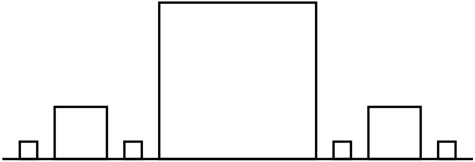

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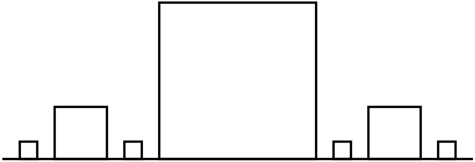

The Cantor function of x, regarded as a function of $z = x + iy$, is then in $A^{(1)}(X) \setminus \mathcal{D}^1_{\mathcal{F}}(X)$.

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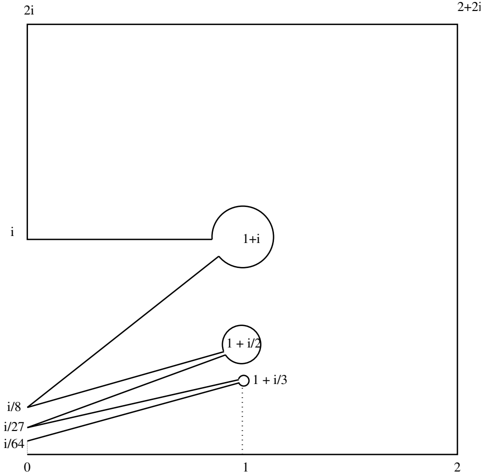

Not to scale

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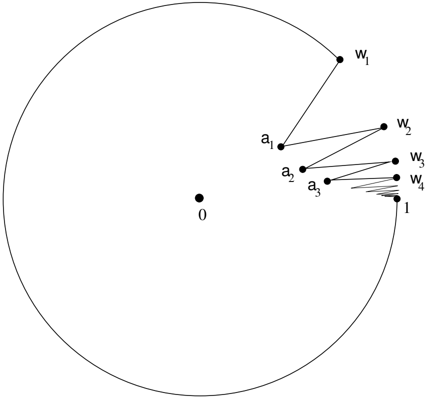

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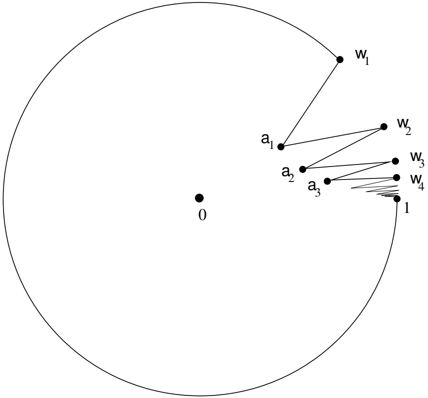

Here, $w_n = e^{i\alpha_n}$ and $a_n = (1 - 4r_n)e^{i\beta_n}$, where $\alpha_n = \dfrac{\pi}{4n^2}$, $r_n = \dfrac{1}{8\sqrt{n}}$ and $\beta_n = (\alpha_n + \alpha_{n+1})/2$.

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