Normed algebras of differentiable functions on compact plane sets

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Abstract

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http://www.maths.nottingham.ac.uk/personal/jff/Beamer

This is joint work with Professor H. Garth Dales (Leeds).

We investigate the completeness and completions of the normed algebras $D^{(1)}(X)$ of continuously complex-differentiable functions on perfect compact plane sets X (discussed by Dales and Davie in 1973).

We solve some problems raised in an earlier paper of Bland and Feinstein (2005) by constructing a variety of compact plane sets X with dense interior such that $D^{(1)}(X)$ is not complete.

We also show that the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

The algebra $D^{(1)}(X)$

Throughout, by **compact plane set** we shall mean an **infinite**, compact subset of $\mathbb{C}$.

Let X be a compact plane set. We denote the set of all continuous, complex-valued functions on X by $C(X)$.

For $f \in C(X)$, we denote the uniform norm of f on a non-empty subset E of X by $|f|_E$.

Definition

Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We then call $f'(a)$ the (**complex**) **derivative** of f at a.

Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

For $f \in D^{(1)}(X)$, set

$$\|f\| = |f|_X + |f'|_X .$$

Then $(D^{(1)}(X), \|\cdot\|)$ is easily seen to be a normed algebra.

The normed algebra $D^{(1)}(X)$ is often incomplete, even for fairly nice X.

Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

The character space of $D^{(1)}(X)$

We now investigate the character space of $D^{(1)}(X)$.

The proof of the next theorem is based on the method used by Jarosz (1997) to determine the character space of $\mathrm{Lip}_{Hol}(X, \alpha)$.

Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \|\cdot\|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.
In particular, every character on $(D^{(1)}(X), \|\cdot\|)$ is continuous.

Proof.

Let ϕ be a character on A, and set $w = \phi(Z)$, where Z is the coordinate functional (restricted to X in this setting).

Then $\phi(Z - w1) = 0$, and so $Z - w1$ is not invertible in A.

Since every rational function with poles off X is in A, it follows that $w \in X$.

We show that ϕ is the point evaluation character at w, ε_w.

To see this, it is sufficient to show that $\ker(\varepsilon_w) \subseteq \ker(\phi)$.

Take $f \in A$ with $f(w) = 0$.

Since

$$\lim_{z \to w,\ z \in X} \frac{f(z)}{z - w} = f'(w)\,,$$

it follows that there is a positive constant C such that, for all $z \in X$, $|f(z)| \leq C|z - w|$.

It is now easy to see that $f^3 = (Z - w1)g$ for a (unique) function $g \in D^{(1)}(X)$ (with $g(w) = g'(w) = 0$).

This gives

$$(\phi(f))^3 = \phi(f^3) = \phi(Z - w1)\phi(g) = 0\,,$$

and so $\phi(f) = 0$.

The result follows. $\square$

Rectifiable paths

We shall assume that the reader is familiar with the elementary results and definitions concerning rectifiable paths including integration of continuous, complex-valued functions along rectifiable paths.

Definition

A **path** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a < b$; γ is a path **from** $\gamma(a)$ **to** $\gamma(b)$ with **endpoints** $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

A **subpath** of γ is then any path obtained by restricting γ to a non-degenerate closed sub-interval of $[a, b]$.

Given $X \subseteq \mathbb{C}$, a **path in** X is a path in $\mathbb{C}$ whose image is a subset of X, and a **Jordan arc in** X is the image of an injective path in X.

The length of a rectifiable path γ will be denoted by $|\gamma|$.

A path in $\mathbb{C}$ is **admissible** if it is rectifiable and has no constant subpaths.

Definition

Let X be a compact plane set.

We say that X is **rectifiably connected** if, for all z and w in X, there is a rectifiable path γ from z to w in X.

Suppose now that X is rectifiably connected.

For z and w in X, we denote the geodesic distance between z and w by $\delta(z, w)$.

We say that such a compact plane set X is **geodesically bounded** if X is is bounded with respect to the metric δ.

We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

Definition

Let X be a compact plane set.

Let $z \in X$. The set X is **regular at** z if there is a constant $k_z > 0$ such that, for every $w \in X$, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k_z|z - w|$.

The set X is **pointwise regular** if X is regular at every point $z \in X$.

The set X is **uniformly regular** if there is one constant $k > 0$ such that, for all z and w in X, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k|z - w|$.

Dales and Davie showed that $D^{(1)}(X)$ is complete whenever X is a finite union of uniformly regular, compact plane sets.

Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

Related spaces

We now recall some related spaces which were discussed in the paper of Bland and Feinstein.

For an open subset U of $\mathbb{C}$, $O(U)$ is the algebra of analytic functions on U.

Now let X be a compact plane set and, throughout, set $U = \operatorname{int} X$.

Then

$$A(X) = \{f \in C(X) : f|_U \in O(U)\}.$$

Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

In this setting, we set

$$\|f\| = |f|_X + |f'|_U \quad (f \in A^{(1)}(X))\,.$$

Then $(A^{(1)}(X), \|\cdot\|)$ is a Banach function algebra on X, and there is an obvious isometric inclusion of $D^{(1)}(X)$ in $A^{(1)}(X)$.

The completion of $D^{(1)}(X)$ is then its closure in $(A^{(1)}(X), \|\cdot\|)$.

The above is only helpful for compact plane sets X such that U is dense in X.

However, Bland and Feinstein introduced some related Banach function algebras $D^{(1)}_{\mathcal{F}}(X)$ defined whenever $\mathcal{F}$ is a suitable family of paths in X.

The completion of $D^{(1)}(X)$

In this section we discuss the completion of $D^{(1)}(X)$, which we denote by $\widetilde{D}^{(1)}(X)$.

As noted by Dales, it follows from an example of Bishop that $\widetilde{D}^{(1)}(X)$ need not be semisimple.

However, from above, we now know some settings where $\widetilde{D}^{(1)}(X)$ is a Banach function algebra.

If X is a compact plane set with dense interior, then we know that $\widetilde{D}^{(1)}(X)$ is simply the closure of $D^{(1)}(X)$ in the Banach function algebra $A^{(1)}(X)$

Our next theorem deals with the situation when the union of the images of all admissible paths in X is dense in X.

Theorem

Let X be a compact plane set such that the union of the images of all admissible paths in X is dense in X.
Then $\widetilde{D}^{(1)}(X)$ is semisimple.

Proof Let $\mathcal{F}$ be the set of all admissible paths in X.
We can show that $\widetilde{D}^{(1)}(X)$ may be identified with the closure of $D^{(1)}(X)$ in the Banach function algebra $D^{(1)}_{\mathcal{F}}(X)$ (mentioned above) .
Thus $\widetilde{D}^{(1)}(X)$ is semisimple. $\square$

Polynomial approximation

The next result is essentially as in Bland and Feinstein (2005).

Theorem

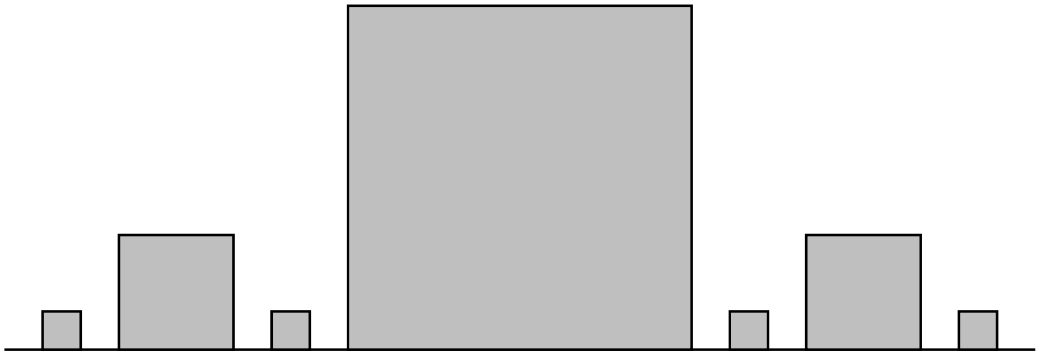
Let X be a polynomially convex, perfect, compact plane set which is geodesically bounded, and let $\mathcal{F}$ be the set of all admissible paths in X. Then the (analytic) polynomials are dense in $D_{\mathcal{F}}^{(1)}(X)$, and so $\widetilde{D}^{(1)}(X) = D_{\mathcal{F}}^{(1)}(X)$.

However, when X has dense interior, the corresponding statement for $A^{(1)}(X)$ is false.

Theorem

There exists a uniformly regular, polynomially convex, compact plane set such that X is geodesically bounded and $\operatorname{int} X$ is dense in X and yet $D^{(1)}(X) = \widetilde{D}^{(1)}(X) = D_{\mathcal{F}}^{(1)}(X) \neq A^{(1)}(X)$.
In particular, the polynomials are not dense in $A^{(1)}(X)$.

An example, based on the Cantor middle thirds set, is illustrated in the following diagram.



The Cantor function of x, regarded as a function of $z = x + \mathrm{i}y$, is then in $A^{(1)}(X) \setminus D^{(1)}_{\mathcal{F}}(X)$.

We now return to the question of the completeness of $(D^{(1)}(X), \|\cdot\|)$.

The following result is due to Dales; it was later rediscovered by Honary and Mahyar.

Theorem

Let X be a perfect, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $A_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z) - f(w)| \le A_z(|f|_X + |f'|_X)|z - w| \,. \tag{1}$$

Note that X need not be connected here. However, the condition implies that X has only finitely many components.

For pointwise regular X, (1) is certainly satisfied, and indeed the $|f|_X$ term may be omitted from the right-hand side of (1).

We now show that this $|f|_X$ term may also be omitted under the weaker assumption that X be connected.

First we require a lemma concerning functions whose derivatives are constantly 0.

Lemma

Let X be a connected, compact plane set for which $(D^{(1)}(X), \|\cdot\|)$ is complete.

Let $f \in D^{(1)}(X)$ be such that $f' = 0$. Then f is a constant.

Proof. Assume towards a contradiction that there exists $f \in D^{(1)}(X)$ such that $f' = 0$ and such that f is not a constant.

We can suppose that $|f|_X = 1$ and that $1 \in f(X)$.

Replacing f by $(1 + f)/2$ if necessary, we may also suppose that 1 is the only value of modulus 1 taken by f on X.

Consideration of the functions $g_n = 1 - f^n$ then quickly leads to a contradiction based on the preceding theorem. $\square$

We are now ready to eliminate the $|f|_X$ term from the right-hand side of equation (1) under the assumption that X is connected.

For convenience, we introduce the following notation.

Let X be a compact plane set, and let $z_0 \in X$. Then we define

$$M_{z_0}^{(1)}(X) = \{f \in D^{(1)}(X) : f(z_0) = 0\}$$

so that $M_{z_0}^{(1)}(X)$ is a maximal ideal in $D^{(1)}(X)$.

Theorem

Let X be a connected, compact plane set for which $(D^{(1)}(X), \|\cdot\|)$ is complete. Let $z_0 \in X$. Then there exists a constant $C_1 > 0$ such that, for all $f \in M_{z_0}^{(1)}(X)$, we have

$$|f|_X \leq C_1 \left|f'\right|_X .$$

Furthermore, there exists another constant $C_2 > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z_0) - f(w)| \leq C_2 \left|f'\right|_X |z_0 - w| . \tag{2}$$

Proof. We shall first prove the existence of the constant C_1.

Assume towards a contradiction that there is a sequence $(f_n) \in M_{z_0}^{(1)}(X)$ such that $|f_n|_X = 1$ for each $n \in \mathbb{N}$, but such that $|f_n'|_X \to 0$ as $n \to \infty$.

We can suppose that $|f_n'|_X \leq 1 \ \ (n \in \mathbb{N})$. Set $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$.

Let $z \in X$. Since $(D^{(1)}(X), \|\cdot\|)$ is complete, there is a constant $C_z > 0$ such that

$$|f(z) - f(w)| \leq C_z(|f|_X + \left|f'\right|_X) |z - w| \leq 2C_z |w - z|$$

for all $f \in \mathcal{F}$ and $w \in X$.

It now follows easily that $\mathcal{F}$ is an equicontinuous family of functions on X.

Certainly $\mathcal{F}$ is bounded in $(C(X), |\cdot|_X)$.

By Ascoli's theorem, $\mathcal{F}$ is relatively compact in $(C(X), |\cdot|_X)$.

By passing to a subsequence, we may suppose that there exists $f \in C(X)$ such that $|f_n - f|_X \to 0$ as $n \to \infty$.

Clearly we have $f(z_0) = 0$ and $|f|_X = 1$. We know that $|f_n'|_X \to 0$ as $n \to \infty$, and so (f_n) is a Cauchy sequence in $(D^{(1)}(X), \|\cdot\|)$.

Since $(D^{(1)}(X), \|\cdot\|)$ is complete, (f_n) is convergent in this space. Clearly $\lim_{n\to\infty} f_n = f$ in $D^{(1)}(X)$, and so $f' = 0$.

By the preceding lemma, f is a constant. But $f(z_0) = 0$, and so $f = 0$, a contradiction of the fact that $|f|_X = 1$.

This proves the existence of the desired constant C_1.

We now set $C_2 = A_{z_0}(1 + C_1)$, where A_{z_0} is the constant from (1).

Equation (2) now follows. $\square$

The above theorem does not hold in the absence of either of the hypotheses that X be connected or that $(D^{(1)}(X), \|\cdot\|)$ be complete.

The following corollary is now immediate.

Corollary

Let X be a connected, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $B_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z) - f(w)| \leq B_z \left|f'\right|_X |z - w| \; . \tag{3}$$

Let X be a polynomially convex, geodesically bounded compact plane set. Then X is connected, and we know that $P_0(X)$ is dense in $(D^{(1)}(X), \|\cdot\|)$.

From this we immediately obtain the following further corollary.

Corollary

Let X be a polynomially convex, geodesically bounded, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $B_z > 0$ such that, for all $p \in P_0(X)$ and all $w \in X$, we have

$$|p(z) - p(w)| \leq B_z \left|p'\right|_X |z - w| \ . \tag{4}$$

We have no counterexample to the following conjecture.

Conjecture 1

Let X be a connected, compact plane set. Then $(D^{(1)}(X), \| \cdot \|)$ is complete if and only if X is pointwise regular.

By a variety of methods, we have proved this conjecture for several classes of compact plane sets.

In particular, the conjecture holds for all rectifiably connected, polynomially convex compact plane sets with empty interior, for all star-shaped, compact plane sets, and for all Jordan arcs in $\mathbb{C}$ (rectifiable or not).

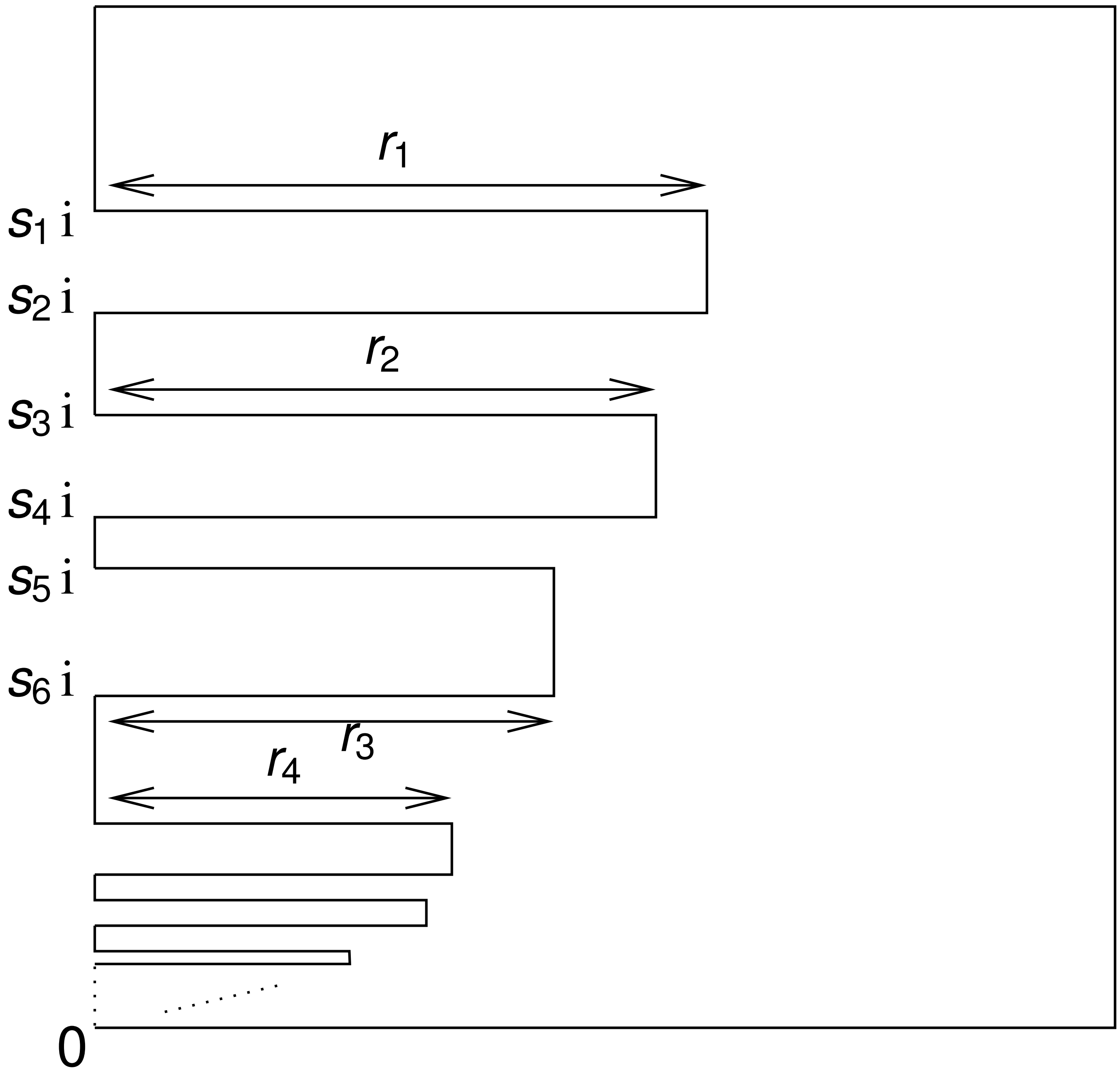
We now give diagrams illustrating some relevant examples.

These examples fully solve a problem raised in the paper of Bland and Feinstein

Example

There exists a polynomially convex, geodesically bounded compact plane set X such that X has dense interior, but $D^{(1)}(X)$ is incomplete.

Indeed, Conjecture 1 holds for all sets of the type shown in the following diagram.



r_1
s_1 i
s_2 i
r_2
s_3 i
s_4 i
s_5 i
s_6 i
r_3
r_4
0

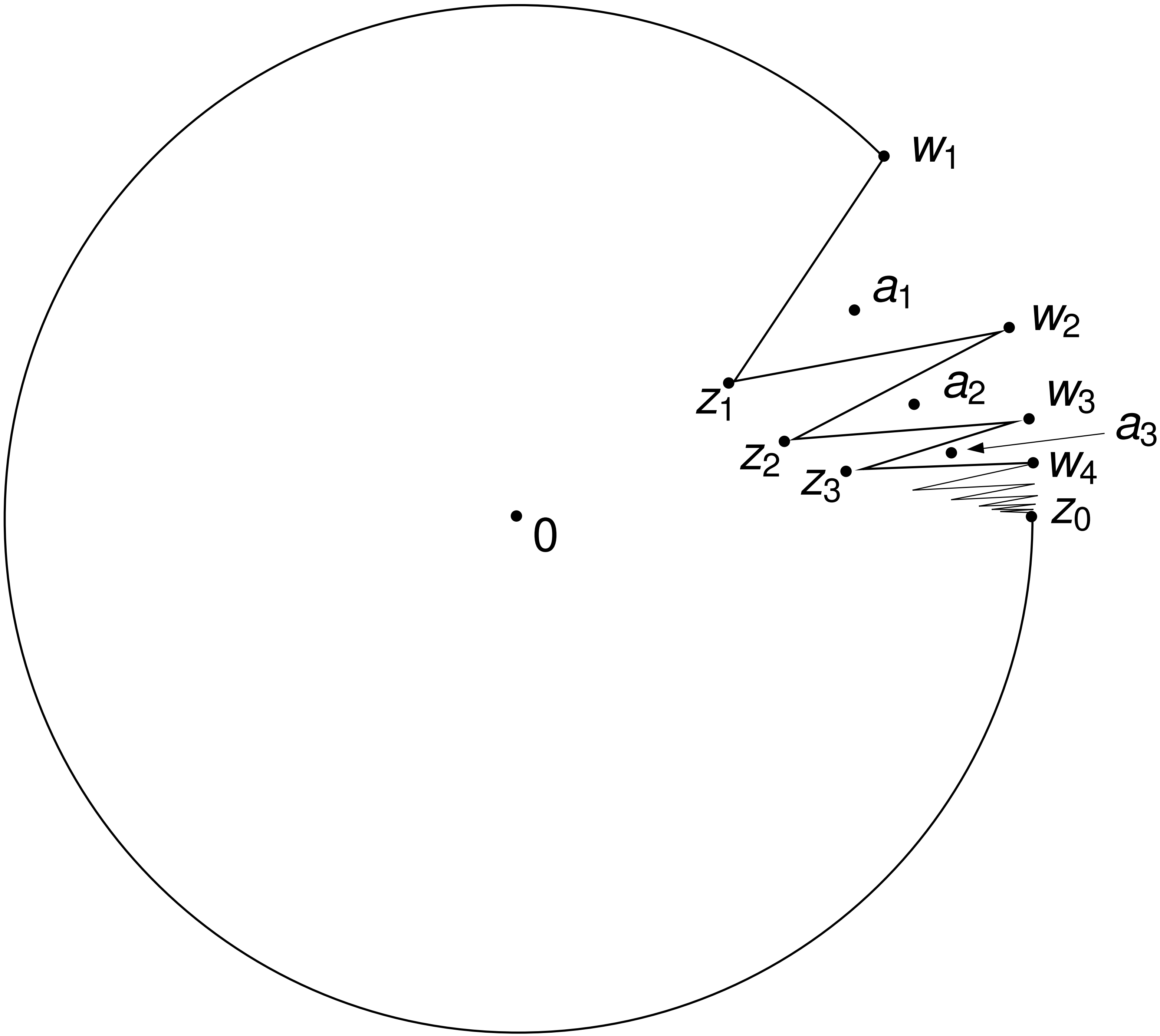
Recall that a compact plane set is **radially self-absorbing** if, for all $r > 1$, $X \subseteq \mathrm{int}\,(rX)$.

Radially self-absorbing, compact plane sets are always star-shaped, polynomially convex, geodesically bounded, and have dense interior.

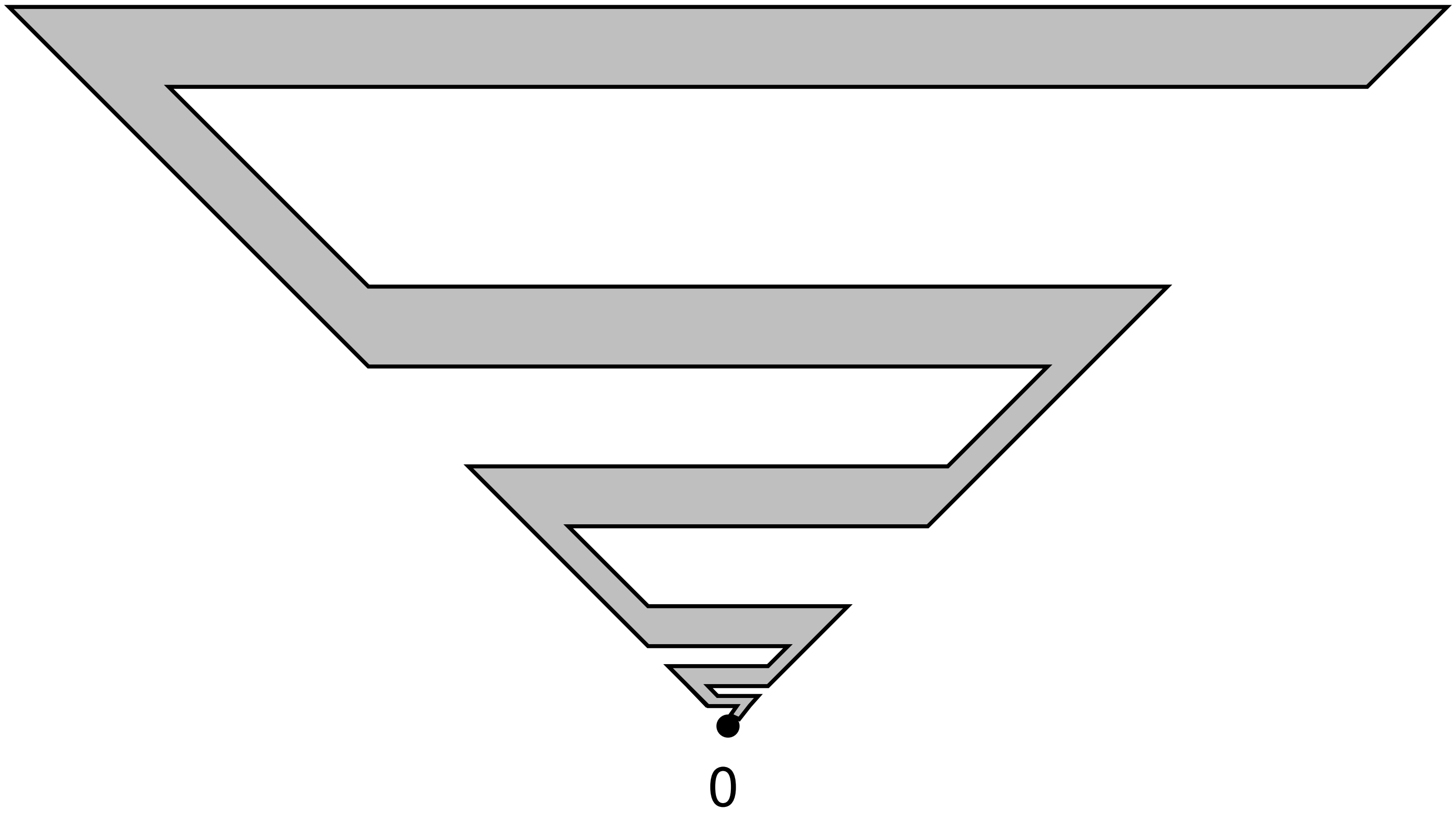
Example

There exists a radially self-absorbing, compact plane set X such that $D^{(1)}(X)$ is incomplete.

This is not hard, given that Conjecture 1 holds for all star-shaped sets.


w_1
a_1
w_2
z_1
a_2
w_3
a_3
z_2
w_4
z_3
z_0
0

We do not know whether or not Conjecture 1 holds for sets of the following type.



Open Questions

We conclude with some open questions concerning perfect, compact plane sets X.

1. Are the rational functions always dense in $(D^{(1)}(X), \| \cdot \|)$? Are the polynomials dense in $(D^{(1)}(X), \| \cdot \|)$ whenever X is polynomially convex?
2. Suppose that $\mathrm{int}\, X$ is dense in X. Is $A^{(1)}(X)$ always natural?
3. Is there a uniformly regular, polynomially convex, compact plane set such that $\mathrm{int}\, X$ is connected and dense in X, and yet

$$\widetilde{D}^{(1)}(X) = D^{(1)}(X) \neq A^{(1)}(X)\,?$$