Completions and completeness of normed algebras of differentiable functions

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Abstract

This is joint work with Professor H. Garth Dales (Leeds).

We investigate the completeness and completions of the normed algebras $D^{(1)}(X)$ of continuously complex-differentiable functions on perfect compact plane sets X.

In particular, we solve a problem raised in an earlier paper of Bland and Feinstein (2005) by constructing a radially self-absorbing compact plane set X such that $D^{(1)}(X)$ is not complete.

An example of Bishop shows that the completion of $D^{(1)}(X)$ need not be semisimple.

We show that the completion of $D^{(1)}(X)$ is semisimple whenever the union of the images of all rectifiable Jordan arcs in X is dense in X.

Introduction

Throughout, by **compact plane set** we shall mean an **infinite**, compact subset of $\mathbb{C}$.

Let X be a perfect, compact plane set, and let $D^{(1)}(X)$ be the normed algebra of all continuously differentiable, complex-valued functions on X (introduced by Dales and Davie in 1973).

In 2005, Bland and Feinstein introduced the notion of $\mathcal{F}$-derivative in order to describe the completions of these spaces under suitable conditions.

Here $\mathcal{F}$ is a suitable set of rectifiable paths lying in a compact plane set X.

We investigate the completeness of the spaces $D^{(1)}(X)$ for compact plane sets X, and discuss their completions.

In particular we answer the problem of Bland and Feinstein concerning radially self-absorbing sets.

The algebra $D^{(1)}(X)$

Let X be a compact plane set. We denote the set of all continuous, complex-valued functions on X by $C(X)$.

For $f \in C(X)$, we denote the uniform norm of f on a non-empty subset E of X by $|f|_E$.

Definition

Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in X} \frac{f(z) - f(a)}{z - a}$$

exists.

We then call $f'(a)$ the (**complex**) **derivative** of f at a.

$D^{(1)}(X)$ continued

Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

For $f \in D^{(1)}(X)$, set

$$\|f\| = |f|_X + |f'|_X \, .$$

Then $(D^{(1)}(X), \|\cdot\|)$ is easily seen to be a normed algebra.

The normed algebra $D^{(1)}(X)$ is often incomplete, even for fairly nice X.

Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

Rectifiable paths

We will assume that the reader is familiar with the elementary results and definitions concerning rectifiable paths including integration of continuous, complex-valued functions along rectifiable paths: for more details see, for example, Apostol's book.

Definition

A **path** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where a and b are real numbers with $a < b$; γ is a path **from** $\gamma(a)$ **to** $\gamma(b)$ with **endpoints** $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

A **subpath** of γ is then any path obtained by restricting γ to a non-degenerate closed sub-interval of $[a, b]$.

Following common usage, we also use γ to denote the image of the path γ.

Paths continued

Definition

Given $X \subseteq \mathbb{C}$, a **path in** X is a path in $\mathbb{C}$ whose image is a subset of X, and a **Jordan arc in** X is an injective path in X.

The length of a rectifiable path γ will be denoted by $|\gamma|$.

A path in $\mathbb{C}$ is **admissible** if it is rectifiable and has no constant subpaths.

Recall that a path $\gamma = \alpha + \mathrm{i}\beta$ (where α and β are real-valued) is rectifiable if and only if both α and β are of bounded variation.

In this case $\int_\gamma f(z)\,\mathrm{d}z$ is defined as a Riemann-Stieltjes integral for all $f \in C(\gamma)$.

Regularity and uniform regularity

Definition

Let X be a compact plane set.

We say that X is **rectifiably connected** if, for all z and w in X, there is a rectifiable path γ from z to w in X.

Suppose now that X is rectifiably connected.

For z and w in X, we denote the geodesic distance between z and w by $\delta(z, w)$.

We say that such a compact plane set X is **geodesically bounded** if X is is bounded with respect to the metric δ.

We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

Regularity continued

Definition

Let X be a compact plane set.

Let $z \in X$. The set X is **regular at** z if there is a constant $k_z > 0$ such that, for every $w \in X$, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k_z|z - w|$.

The set X is **pointwise regular** if X is regular at every point $z \in X$.

The set X is **uniformly regular** if there is one constant $k > 0$ such that, for all z and w in X, there is a rectifiable path γ from z to w in X with $|\gamma| \leq k|z - w|$.

Dales and Davie showed that $D^{(1)}(X)$ is complete whenever X is a finite union of uniformly regular, compact plane sets.

Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

Related spaces

We now recall some related spaces which were discussed in the paper of Bland and Feinstein.

For an open subset U of $\mathbb{C}$, $O(U)$ is the algebra of analytic functions on U.

Now let X be a compact plane set and, throughout, set $U = \operatorname{int} X$.

Then

$$A(X) = \{f \in C(X) : f|_U \in O(U)\}.$$

Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

Related spaces continued

In this setting, we set

$$\|f\| = |f|_X + |f'|_U \quad (f \in A^{(1)}(X))\,.$$

Then $(A^{(1)}(X), \|\cdot\|)$ is a Banach function algebra on X, and there is an obvious isometric inclusion of $D^{(1)}(X)$ in $A^{(1)}(X)$.

The completion of $D^{(1)}(X)$ is then its closure in $(A^{(1)}(X), \|\cdot\|)$.

The above is only helpful for compact plane sets X such that U is dense in X.

This is too restrictive for our purposes, and instead we shall mostly work with the larger class of compact plane sets X for which the union of the images of all admissible rectifiable paths in X is dense in X.

Related spaces continued

We begin by defining a new term, 'effective', which is a modification of the term 'useful' introduced by Bland and Feinstein.

Definition

Let X be a compact plane set, and let $\mathcal{F}$ be a family of paths in X.

Then $\mathcal{F}$ is **effective** if each subpath of a path in $\mathcal{F}$ belongs to $\mathcal{F}$, if each path in $\mathcal{F}$ is rectifiable and non-constant, and the union of the images of the paths in $\mathcal{F}$ is dense in X.

Note that, if $\mathcal{F}$ is effective, then every path in $\mathcal{F}$ is admissible.

We will often take $\mathcal{F}$ to be the set of all admissible paths in X.

In this case, $\mathcal{F}$ is effective if and only if the union of the images of all admissible paths in X is dense in X.

Related spaces continued

The next few definitions and results are essentially as in the paper of Bland and Feinstein, although some proofs require a little more work in the setting of effective families.

Recall that the endpoints of a path γ are denoted by γ^- and γ^+.

Definition

Let $\mathcal{F}$ be a family of rectifiable paths in a compact plane set X.

For $f \in C(X)$, we say that $g \in C(X)$ is an $\mathcal{F}$-**derivative** of f if, for all $\gamma \in \mathcal{F}$, we have

$$\int_\gamma g(z)\,\mathrm{d}z = f(\gamma^+) - f(\gamma^-).$$

We define

$$\mathcal{D}^1_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

Related spaces continued

Lemma

Let X be a perfect, compact plane set and let $\mathcal{F}$ be a set of rectifiable paths in X.

(a) *Let $f \in D^{(1)}(X)$. Then the usual derivative f' is also an $\mathcal{F}$-derivative of f.*

(b) *Let $f_1, f_2 \in D^{(1)}(X)$. Then $f_1 f_2' + f_1' f_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$.*

We are mostly interested in the case where $\mathcal{F}$ is effective.

In this case $\mathcal{F}$-derivatives are unique.

Theorem

Let X be a compact plane set, let $\mathcal{F}$ be an effective family of paths in X, and let $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$ have $\mathcal{F}$-derivatives g_1, g_2, respectively. Then $f_1 g_2 + g_1 f_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$.

Related spaces continued

As observed above, if $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ by f'.

Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

For $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, set $\|f\| = |f|_X + |f'|_X$.

Theorem

Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

Then $(\mathcal{D}^1_{\mathcal{F}}(X), \|.\|)$ is a Banach function algebra containing $D^{(1)}(X)$ as a subalgebra.

Note that the inclusion map from $D^{(1)}(X)$ to $\mathcal{D}^1_{\mathcal{F}}(X)$ is obviously isometric here.

The completion of $D^{(1)}(X)$

In this section we discuss the completion of $D^{(1)}(X)$, which we denote by $\widetilde{D}^{(1)}(X)$.

Recall that, for a Banach algebra A, the semi-direct product $A \ltimes A$ (as opposed to $A \rtimes A$) is the Banach algebra whose underlying Banach space is the ℓ_1 direct sum $A \oplus_1 A$, and whose product is defined by

$$(a,b)(a',b') = (aa', ab' + ba') \quad (a,b,a',b' \in A)\,.$$

Proposition

(Dales) *Let X be a perfect, compact plane set.*

Then the map ι defined by $f \mapsto (f, f')$ is an isometric algebra embedding of $D^{(1)}(X)$ in the semi-direct product $C(X) \ltimes C(X)$, and $\widetilde{D}^{(1)}(X)$ may be identified with the closure in $C(X) \ltimes C(X)$ of $\iota(D^{(1)}(X))$.

Completion of $D^{(1)}(X)$ continued

We now give an example to show that $\widetilde{D}^{(1)}(X)$ need not be semisimple.

Example

(Dales) Bishop (1958) gave an example of a Jordan arc J in the plane with the property that the image under the above embedding ι of the set of polynomial functions is dense in $C(J) \ltimes C(J)$.

It follows immediately from this that $\widetilde{D}^{(1)}(J)$ is equal to $C(J) \ltimes C(J)$.

In particular, this completion is not semisimple.

Bishop's arc J has no rectifiable subarcs.

This is not too surprising in view of our next result.

Completion of $D^{(1)}(X)$ continued

Theorem

Let X be a compact plane set such that the union of the images of all admissible rectifiable paths in X is dense in X.
Then $\widetilde{D}^{(1)}(X)$ is semisimple.

Proof Let $\mathcal{F}$ be the set of all admissible paths in X. Then $\mathcal{F}$ is effective.

We know that $\mathcal{D}^1_{\mathcal{F}}(X)$ is a Banach function algebra, and that we can regard $\widetilde{D}^{(1)}(X)$ as the closure of $D^{(1)}(X)$ in $\mathcal{D}^1_{\mathcal{F}}(X)$.

Thus $\widetilde{D}^{(1)}(X)$ is semisimple. $\square$

For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.

In general there are easy examples where $\mathcal{F}$ is effective but $\widetilde{D}^{(1)}(X) \neq \mathcal{D}^1_{\mathcal{F}}(X)$.

Completeness of $D^{(1)}(X)$

We now return to the question of the completeness of $D^{(1)}(X)$.

The following result, due to Dales, was rediscovered by Honary and Mahyar (1999).

Theorem

Let X be a perfect, compact plane set. Then $D^{(1)}(X)$ is complete if and only if, for each $z \in X$, there exists $A_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X \setminus \{z\}$,

$$\frac{|f(w) - f(z)|}{|w - z|} \leq A_z(|f|_X + |f'|_X)\,.$$

Note that X need not be connected here.

For pointwise regular X this condition is certainly satisfied, and indeed the $|f|_X$ term may be omitted from the right hand side of the inequality above. This is part of the motivation for the next theorem.

Completeness continued

Theorem

Let X be a polynomially convex, perfect, compact plane set which is geodesically bounded, and let $\mathcal{F}$ be the set of all admissible rectifiable paths in X. Then the following statements are equivalent:

- **(a)** $D^{(1)}(X)$ *is complete;*
- **(b)** $D^{(1)}(X) = \mathcal{D}^1_{\mathcal{F}}(X)$;
- **(c)** *for each $z \in X$, there exists $C_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X \setminus \{z\}$, we have*

$$\frac{|f(w) - f(z)|}{|w - z|} \leq C_z |f'|_X \, ; \tag{1}$$

In fact (as shown by Bland and Feinstein) the (analytic) polynomials are dense in $\mathcal{D}^1_{\mathcal{F}}(X)$ in this setting, and so it is enough to check condition (1) for polynomials.

Completeness continued

In the case where, additionally, X has dense interior, one might suspect that a similar argument would prove that the polynomials are dense in $A^{(1)}(X)$.

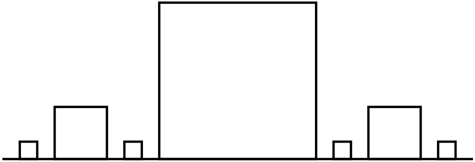
This is, however, false.

Theorem

There exists a uniformly regular, polynomially convex, compact plane set such that X is geodesically bounded and $\operatorname{int} X$ is dense in X and yet $D^{(1)}(X) = \mathcal{D}^1_{\mathcal{F}}(X) \neq A^{(1)}(X)$.
In particular, the polynomials are not dense in $A^{(1)}(X)$.

Completeness continued

An example, based on the Cantor middle thirds set, is illustrated in the following diagram.



The Cantor function of x, regarded as a function of $z = x + \mathrm{i}y$, is then in $A^{(1)}(X) \setminus \mathcal{D}^1_{\mathcal{F}}(X)$.

Completeness continued

We now give diagrams illustrating two examples to show that $D^{(1)}(X)$ need not be complete when X is polynomially convex, geodesically bounded and has dense interior.

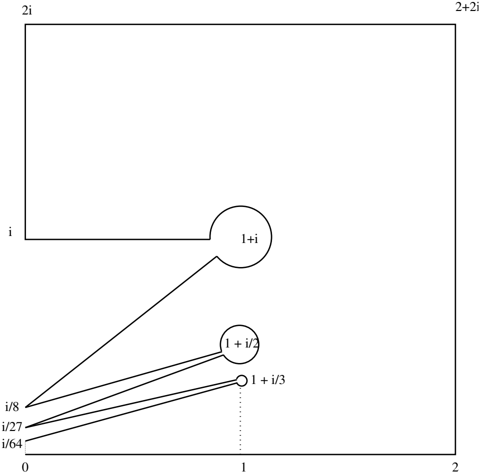
These examples fully solve Problem 6.2 raised in the paper of Bland and Feinstein

Example

There exists a polynomially convex, geodesically bounded compact plane set X such that X has dense interior, but $D^{(1)}(X)$ is incomplete.

Completeness continued

Here is a diagram of such an example.



Not to scale

Completeness continued

Recall that a compact plane set is **radially self-absorbing** if, for all $r > 1$, $X \subseteq \mathrm{int}\,(rX)$.

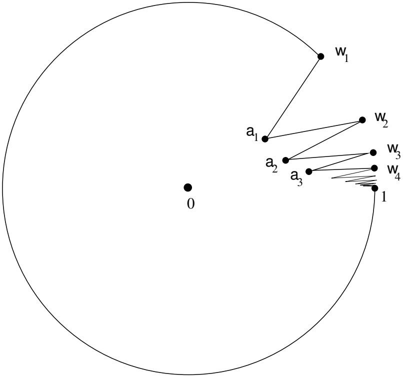
Radially self-absorbing sets are always polynomially convex and geodesically bounded, and have dense interior.

Example

There exists a radially self-absorbing compact plane set X such that $D^{(1)}(X)$ is incomplete.

Completeness continued

Here is a diagram of such an example.



Here, $w_n = \mathrm{e}^{\mathrm{i}\alpha_n}$ and $a_n = (1 - 4r_n)\mathrm{e}^{\mathrm{i}\beta_n}$, where $\alpha_n = \dfrac{\pi}{4n^2}$, $r_n = \dfrac{1}{8\sqrt{n}}$ and $\beta_n = (\alpha_n + \alpha_{n+1})/2$.

Completeness continued

We do not know of an example of a connected, compact plane set X such that X is not pointwise regular, and yet $D^{(1)}(X)$ is complete.

We conclude with a partial result which eliminates a large class of compact plane sets, including all rectifiable Jordan arcs.

Theorem

Let X be a simply connected, geodesically bounded, compact plane set with empty interior.

If X is not pointwise-regular, then $D^{(1)}(X)$ is incomplete.