Completions and completeness of normed algebras of differentiable functions

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An example of Bishop shows that the completion of $D^{(1)}(X)$ need not be semisimple.

We show that the completion of $D^{(1)}(X)$ is semisimple whenever the union of the images of all rectifiable Jordan arcs in X is dense in X.

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In particular we answer the problem of Bland and Feinstein concerning radially self-absorbing sets.

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Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

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We then call $f'(a)$ the (**complex**) **derivative** of f at a.

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Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

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The normed algebra $D^{(1)}(X)$ is often incomplete, even for fairly nice X.

Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

Rectifiable paths

We will assume that the reader is familiar with the elementary results and definitions concerning rectifiable paths including integration of continuous, complex-valued functions along rectifiable paths: for more details see, for example, Apostol's book.

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Following common usage, we also use γ to denote the image of the path γ.

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In this case $\int_\gamma f(z)\,\mathrm{d}z$ is defined as a Riemann-Stieltjes integral for all $f \in C(\gamma)$.

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We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

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Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

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Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

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This is too restrictive for our purposes, and instead we shall mostly work with the larger class of compact plane sets X for which the union of the images of all admissible rectifiable paths in X is dense in X.

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In this case, $\mathcal{F}$ is effective if and only if the union of the images of all admissible paths in X is dense in X.

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We define

$$\mathcal{D}^1_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

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(b) *Let $f_1, f_2 \in D^{(1)}(X)$. Then $f_1 f_2' + f_1' f_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$.*

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Theorem

Let X be a compact plane set, let $\mathcal{F}$ be an effective family of paths in X, and let $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$ have $\mathcal{F}$-derivatives g_1, g_2, respectively.

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(b) *Let $f_1, f_2 \in D^{(1)}(X)$. Then $f_1 f_2' + f_1' f_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$.*

We are mostly interested in the case where $\mathcal{F}$ is effective.

In this case $\mathcal{F}$-derivatives are unique.

Theorem

Let X be a compact plane set, let $\mathcal{F}$ be an effective family of paths in X, and let $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$ have $\mathcal{F}$-derivatives g_1, g_2, respectively. Then $f_1 g_2 + g_1 f_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$.

Related spaces continued

As observed above, if $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ by f'.

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Note that the inclusion map from $D^{(1)}(X)$ to $\mathcal{D}^1_{\mathcal{F}}(X)$ is obviously isometric here.

The completion of $D^{(1)}(X)$

In this section we discuss the completion of $D^{(1)}(X)$, which we denote by $\widetilde{D}^{(1)}(X)$.

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Proposition

(Dales) *Let X be a perfect, compact plane set.*

Then the map ι defined by $f \mapsto (f, f')$ is an isometric algebra embedding of $D^{(1)}(X)$ in the semi-direct product $C(X) \ltimes C(X)$, and $\widetilde{D}^{(1)}(X)$ may be identified with the closure in $C(X) \ltimes C(X)$ of $\iota(D^{(1)}(X))$.

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Bishop's arc J has no rectifiable subarcs.

This is not too surprising in view of our next result.

Completion of $D^{(1)}(X)$ continued

Theorem

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Proof Let $\mathcal{F}$ be the set of all admissible paths in X. Then $\mathcal{F}$ is effective. We know that $\mathcal{D}^1_{\mathcal{F}}(X)$ is a Banach function algebra, and that we can regard $\widetilde{D}^{(1)}(X)$ as the closure of $D^{(1)}(X)$ in $\mathcal{D}^1_{\mathcal{F}}(X)$.

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}^1_{\mathcal{F}}(X)$.
In general there are easy examples where $\mathcal{F}$ is effective but $\widetilde{D}^{(1)}(X) \neq \mathcal{D}^1_{\mathcal{F}}(X)$.

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Note that X need not be connected here.

For pointwise regular X this condition is certainly satisfied, and indeed the $|f|_X$ term may be omitted from the right hand side of the inequality above. This is part of the motivation for the next theorem.

Completeness continued

Theorem

Let X be a polynomially convex, perfect, compact plane set which is geodesically bounded, and let $\mathcal{F}$ be the set of all admissible rectifiable paths in X. Then the following statements are equivalent:

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- **(a)** $D^{(1)}(X)$ *is complete;*
- **(b)** $D^{(1)}(X) = \mathcal{D}^1_{\mathcal{F}}(X)$;
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In fact (as shown by Bland and Feinstein) the (analytic) polynomials are dense in $\mathcal{D}^1_{\mathcal{F}}(X)$ in this setting, and so it is enough to check condition (1) for polynomials.

Completeness continued

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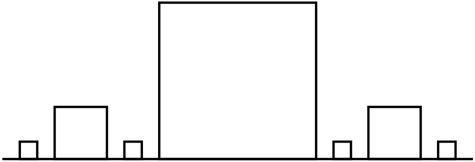

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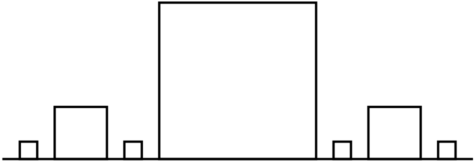

The Cantor function of x, regarded as a function of $z = x + \mathrm{i}y$, is then in $A^{(1)}(X) \setminus \mathcal{D}^1_{\mathcal{F}}(X)$.

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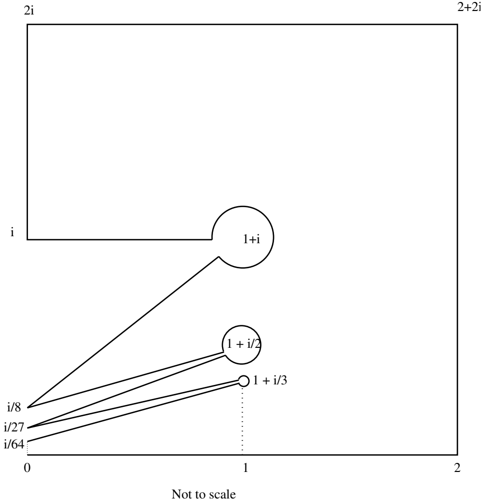

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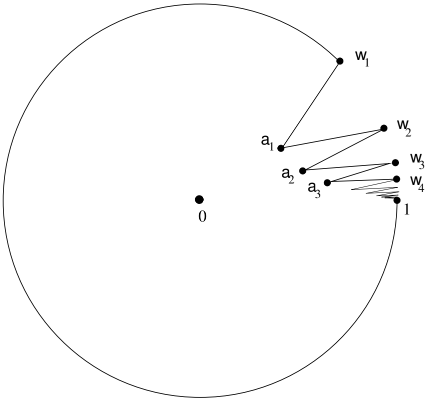

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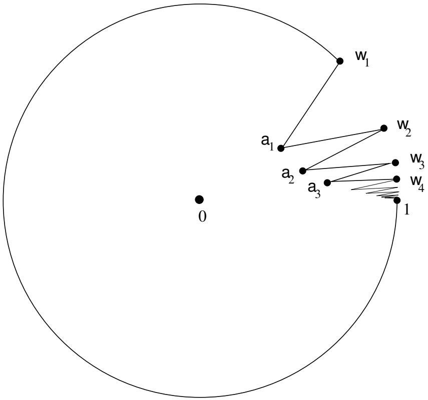

Here, $w_n = e^{i\alpha_n}$ and $a_n = (1 - 4r_n)e^{i\beta_n}$, where $\alpha_n = \dfrac{\pi}{4n^2}$, $r_n = \dfrac{1}{8\sqrt{n}}$ and $\beta_n = (\alpha_n + \alpha_{n+1})/2$.

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Theorem

Let X be a simply connected, geodesically bounded, compact plane set with empty interior.

If X is not pointwise-regular, then $D^{(1)}(X)$ is incomplete.