

Beyond Infinity?

Joel Feinstein

School of Mathematical Sciences
University of Nottingham

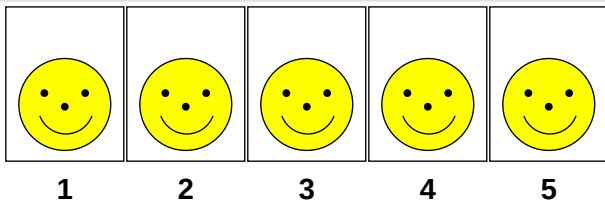
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The serious mathematics behind this talk
is due to the great mathematicians
David Hilbert (1862–1943) and
Georg Cantor (1845–1918)

Hilbert's Grand Hotel

This is a story about Hilbert's Grand Hotel ...
and how Dave, the hotel manager, attempts to deal with the problems
arising from its over-popularity.

Most hotels only have a finite number of rooms.

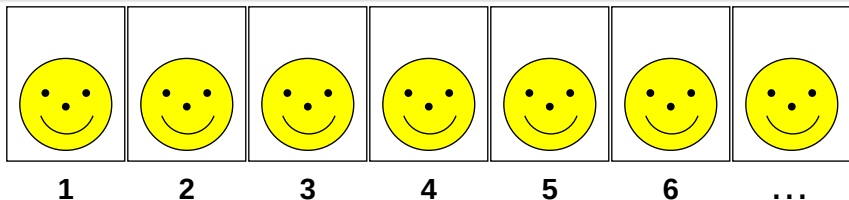


When every room is full, they can not take any more guests.

The Grand Hotel

Hilbert's Grand Hotel is different!

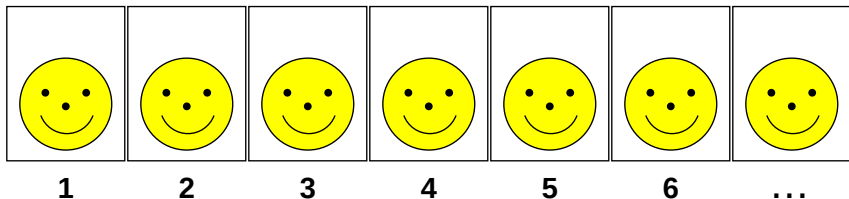
It has infinitely many rooms, numbered 1, 2, 3, 4,



Even so, it is so popular that quite often every room has a guest in it.

What can Dave, the hotel manager, do if another guest turns up?

Room for one more!



One day, when every room had a guest in it, one more guest arrived.

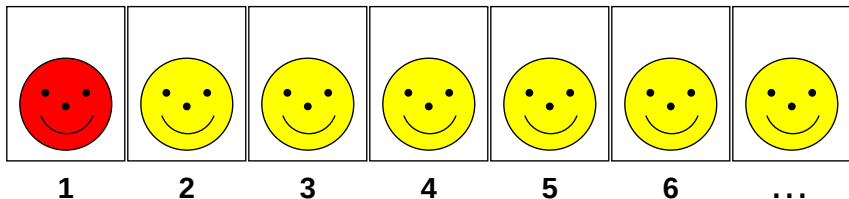


“No problem!” said Dave.

“A minor inconvenience for each guest, that’s all.”

What did Dave do?

Room for one more!



Dave asked the guests in each room to move one room to the right, i.e., for each guest to move to the room whose number was one more than the one they were currently in.

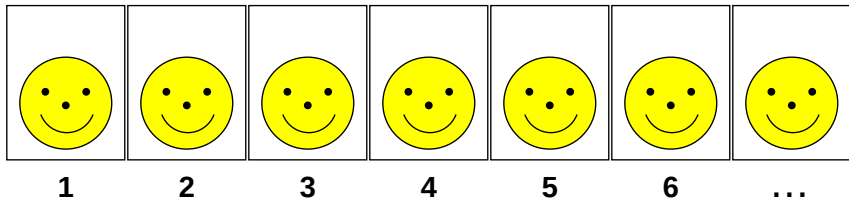
So, in particular, the guest in room 7 moved to room 8.

The guest in room 6 moved to room 7.

The guest in room 5 moved to room 6, etc.

This left room 1 empty, and the new guest moved in there.

Room for more?



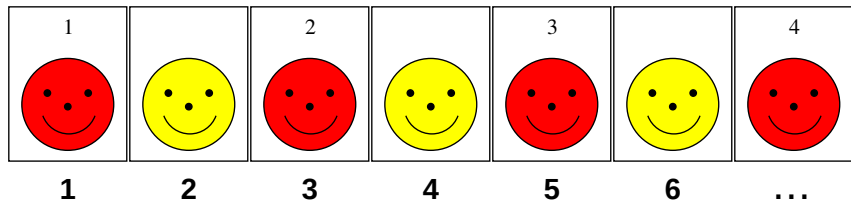
A few days later, with all the rooms full as usual, a bus turned up which had infinitely many passengers on board. Their shirts were numbered 1, 2, 3, 4,



Dave scratched his head for a bit, and then announced “No problem!”
“A minor inconvenience for each guest, that’s all.”

What did Dave do?

Room for more?



Dave asked each guest to move to the room whose number was twice the number of the room they were currently in.

So, in particular, the guests in room 7, 6, 5 and 4 moved to rooms 14, 12, 10 and 8 respectively.

The guest in room 3 moved to room 6.

The guest in room 2 moved to room 4.

The guest in room 1 moved to room 2.

This left rooms 1, 3, 5, 7, ... empty, and the new guests moved in.

Room for even more?

A week later, another bus turned up with infinitely many passengers. This time their shirts were labelled with the positive fractions in their lowest terms: $\frac{1}{2}$, $\frac{11}{6}$, $\frac{3}{1000}$, etc., and there was one passenger for each positive fraction.

Dave scratched his head for quite a while, but then he smiled and said “No problem!”

“A minor inconvenience for each guest, that’s all.”

Dave started, as before, by freeing up rooms 1, 3, 5, 7,

Then he said “Right! If your shirt is labelled m/n , then you can stay in room $3^m 5^n$.” [In an earlier version, Dave used $2^m 3^n - 1$, as mentioned in the audio podcast.]

At a stroke, Dave had not only managed to fit in all his new guests, he had even managed to free up infinitely many rooms, with minimal inconvenience to his guests!

Room for even more?

The next week, when the hotel was full again, infinitely many buses turned up. The buses were numbered 1, 2, 3, 4,

Each of the buses had infinitely many passengers!
Passenger n from bus m had a shirt labelled (m, n) .

This time Dave didn't even have to think.
"I can do this exactly the same way I did the fractions!"
Again he started by freeing up rooms 1, 3, 5, 7,
Then he put passenger (m, n) in room $3^m 5^n$.

The next week, though, Dave finally had to admit defeat.

Too many more?

The bus looked innocuous enough: it belonged to a firm called **Cannes Tours**.

Of course it had infinitely many passengers on it.

There was one passenger for every real number between 0 and 0.5, each of them wearing a shirt labelled with the appropriate decimal expansion.

Where there was a choice, the expansion always ended in recurring 0's rather than recurring 9's.

Dave thought and thought, but eventually he smiled ruefully.

"I'm sorry!" said Dave. "You'll have to try the **Hotel Uncountable** round the corner. We can't fit you in here."

Was Dave right?

Cannes Tours diagonalization argument

Suppose, for contradiction, that Dave has managed to find a way to fit in all his guests.

We define the following numbers b_n , all of which are either 3 or 4.

If room n does not have a guest from Cannes Tours in it, we set $b_n = 3$. Otherwise, room n **does** have a guest from Cannes Tours.

We know that this guest is labelled with a decimal expansion, say $0.a_1a_2a_3a_4\dots$

We set $b_n = 3$ if $a_n \neq 3$, while if $a_n = 3$ we set $b_n = 4$.

So b_n is 3 or 4, and $b_n \neq a_n$.

In particular, if a guest from Cannes Tours is in room n , then their decimal expansion does not have b_n in the n th place.

Cantor's diagonalization argument (conclusion)

From previous slide

Whenever a guest from Cannes Tours is in room n , then their decimal expansion does not have b_n in the n th place.

We now have a sequence $b_1, b_2, b_3, \dots$ of 3's and 4's.

Consider the number $x = 0.b_1b_2b_3\dots$

The number x is obviously a real number between 0 and 0.5.

Note that this expansion doesn't end in recurring 9's or recurring 0's.

This means one of the Cannes Tours guests, namely guest x , is labelled with the expansion $0.b_1b_2b_3\dots$

Guest x must be in one of the rooms, say room n .

But then guest x from Cannes Tours is in room n , and has a decimal expansion with b_n in the n th place, which contradicts our choice of b_n above.

Beyond infinity?

This contradiction shows that, even if the hotel starts off empty, there really is not enough room for all of the guests from Cannes Tours.

Fortunately the **Hotel Uncountable** has a room for every real number! One day, though, even the **Hotel Uncountable** couldn't cope ... but that's another story!

THE END

