Beyond Infinity?

Joel Feinstein

School of Mathematical Sciences
University of Nottingham

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The serious mathematics behind this talk
is due to the great mathematicians
David Hilbert (1862–1943) and
Georg Cantor (1845–1918)

Hilbert's Grand Hotel

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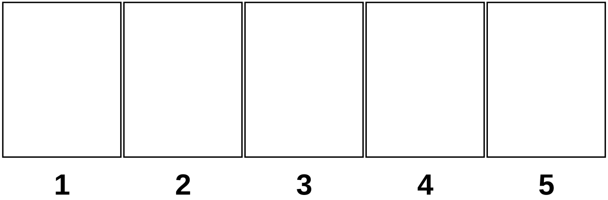

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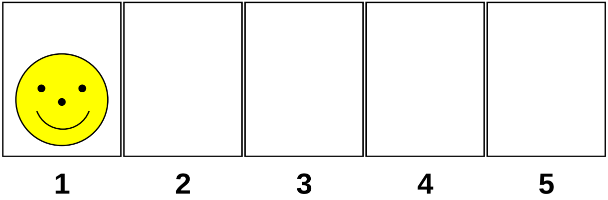

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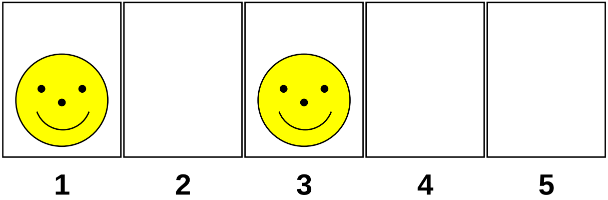

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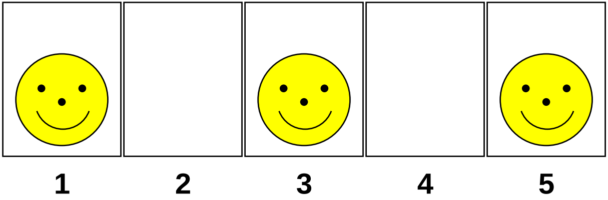

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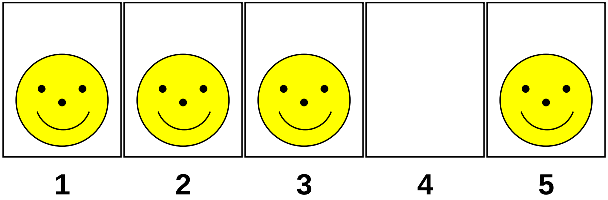

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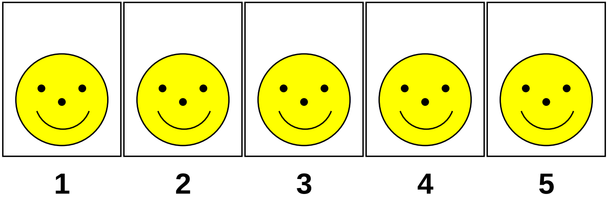

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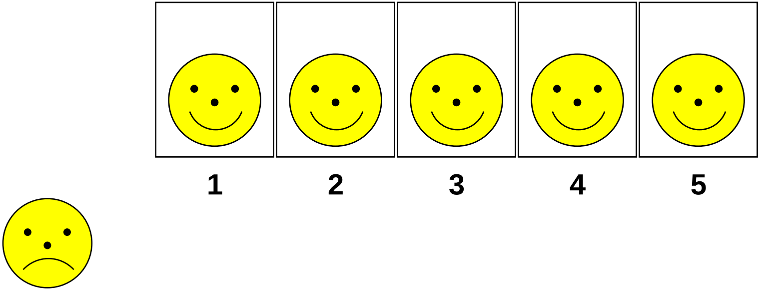

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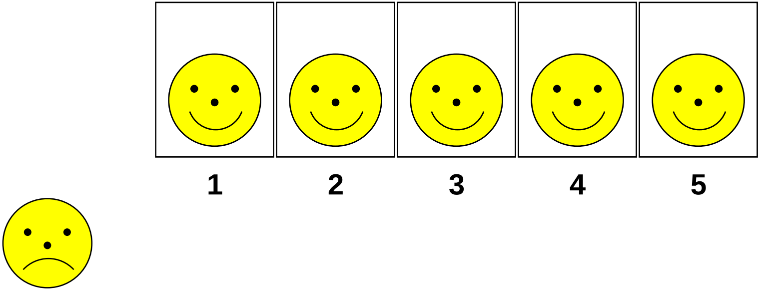

When every room is full, they can not take any more guests.

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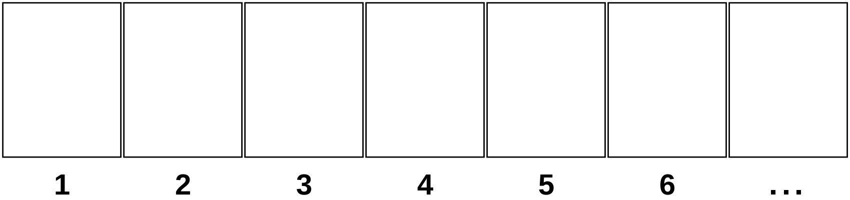

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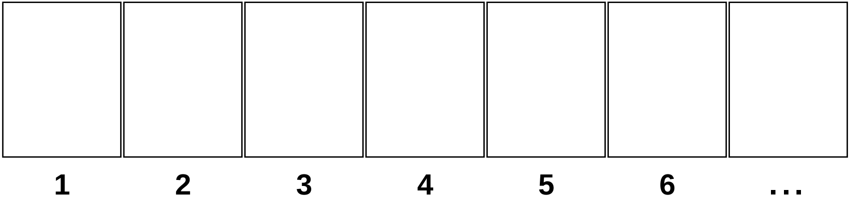

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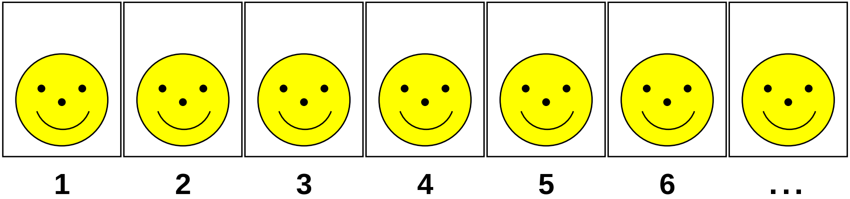

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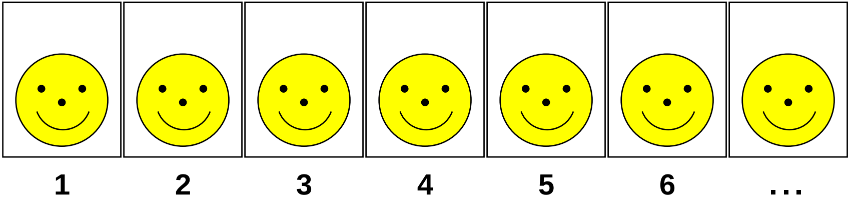

Even so, it is so popular that quite often every room has a guest in it.

What can Dave, the hotel manager, do if another guest turns up?

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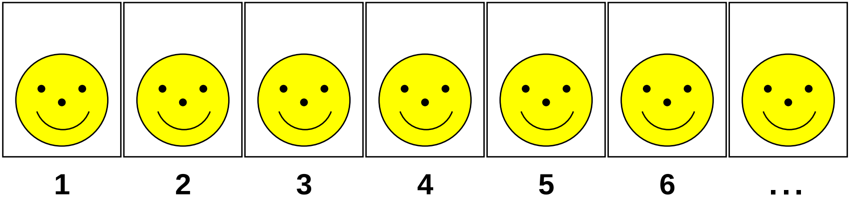

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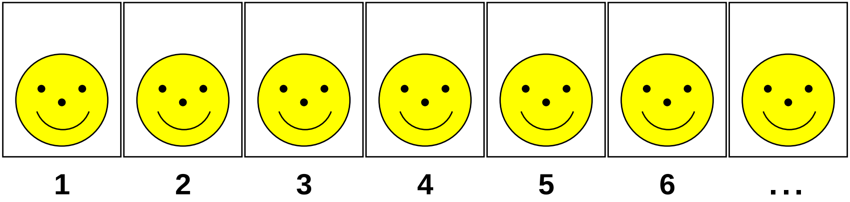

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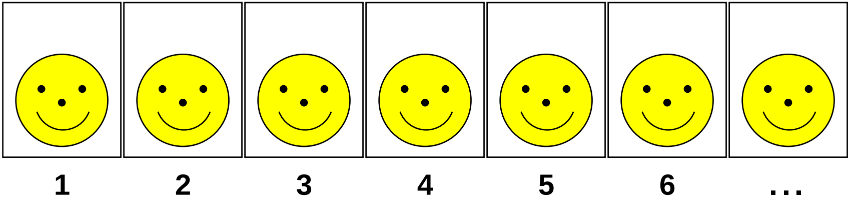

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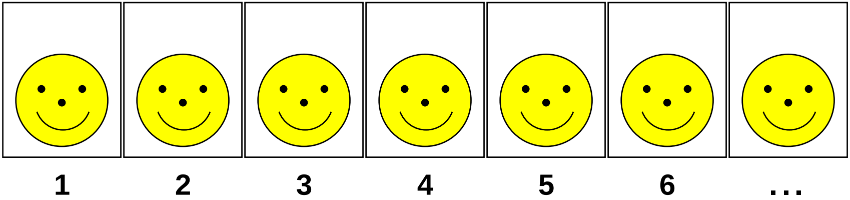

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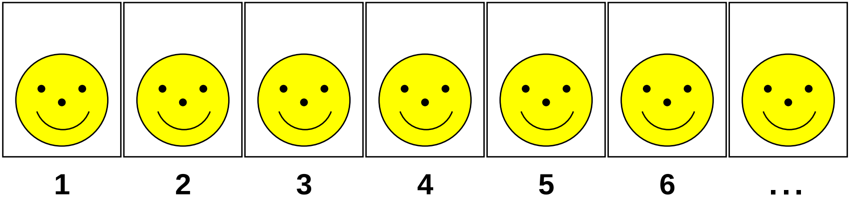

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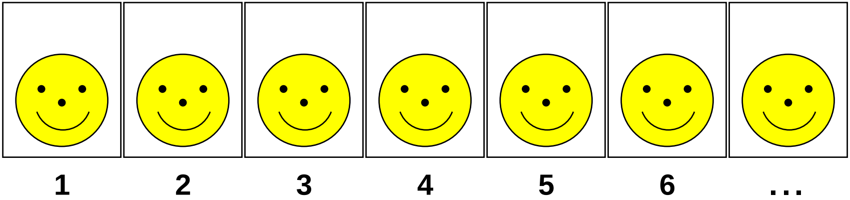

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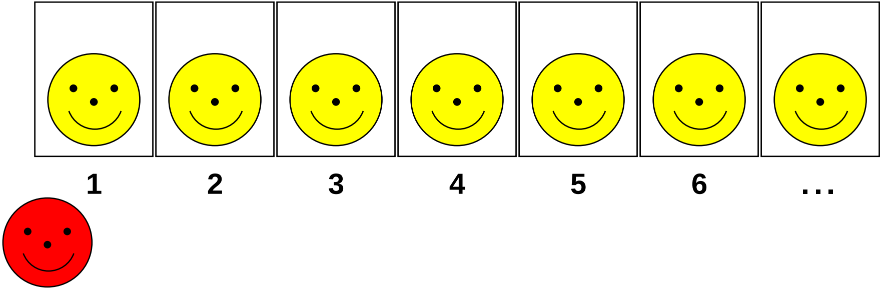

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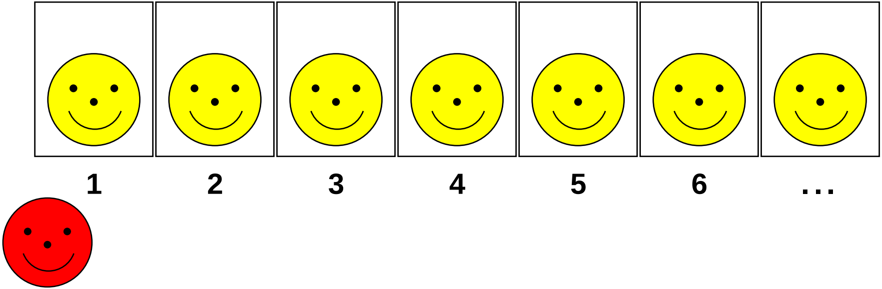

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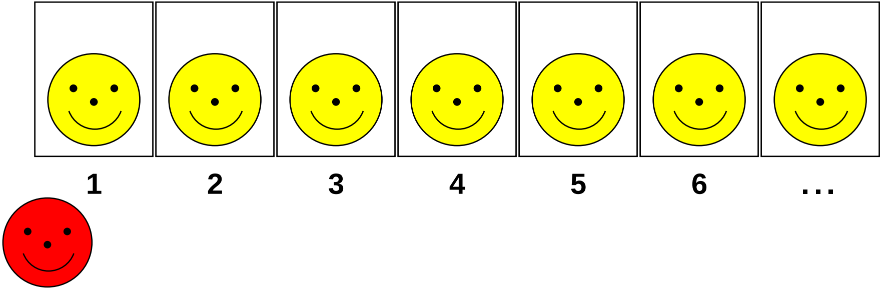

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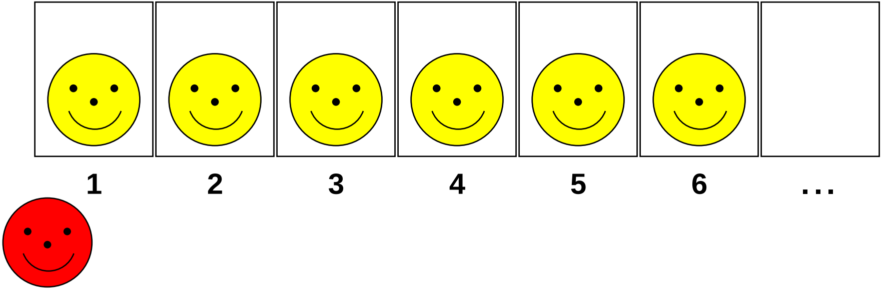

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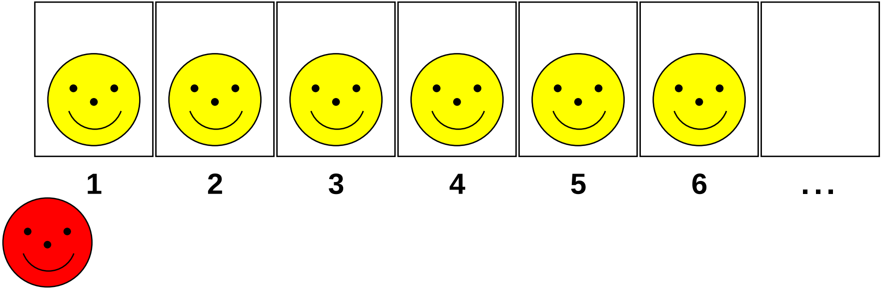

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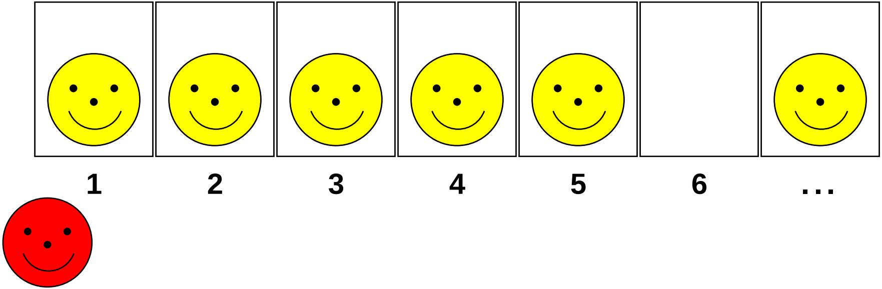

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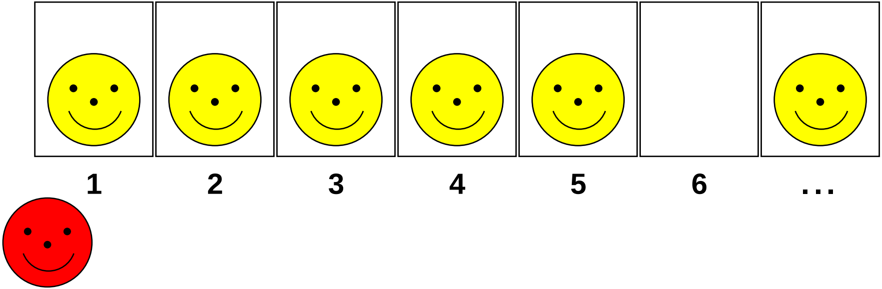

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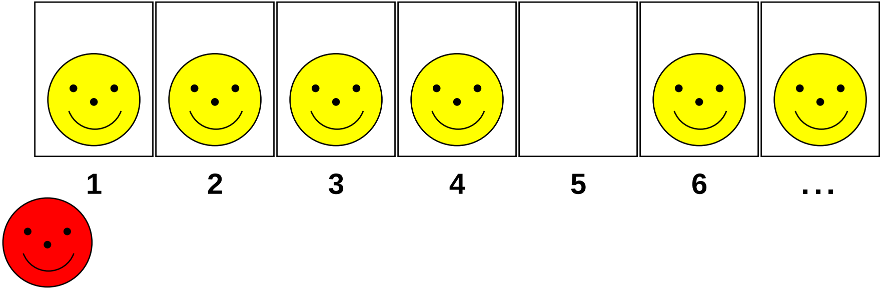

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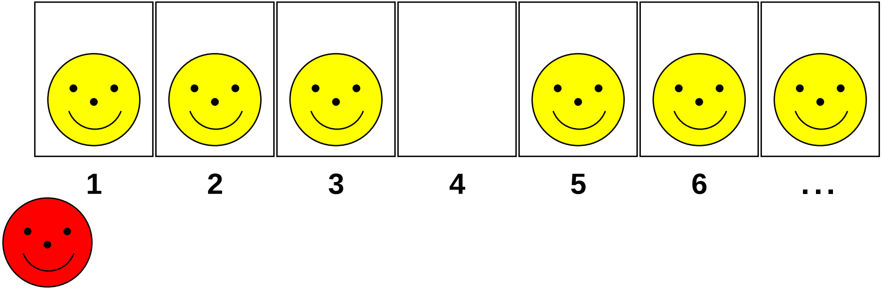

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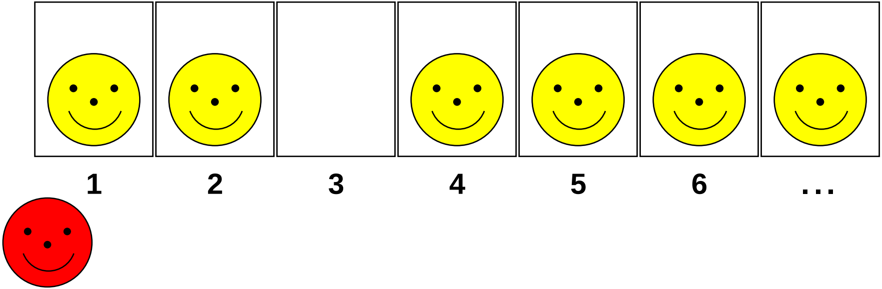

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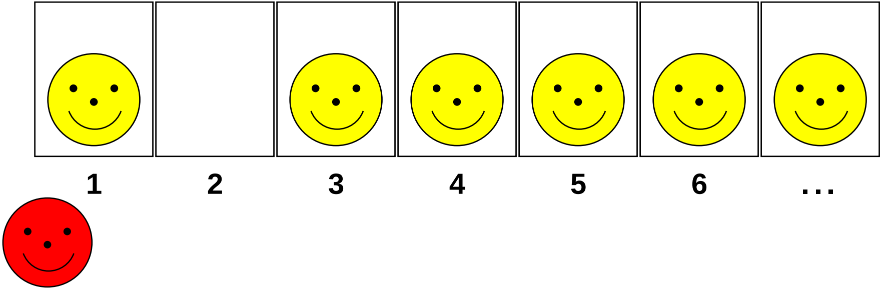

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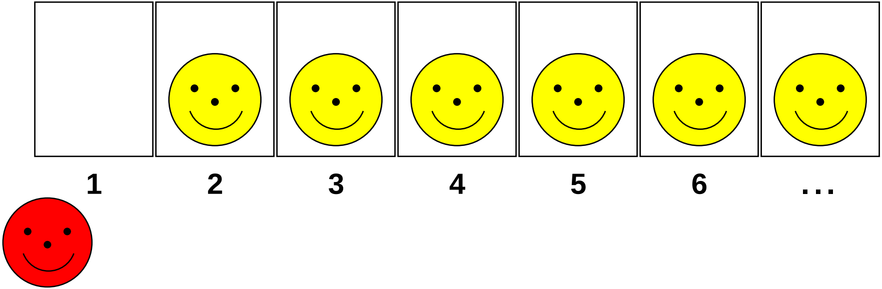

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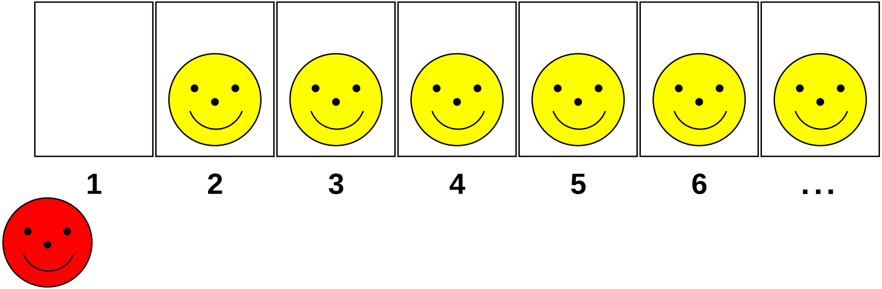

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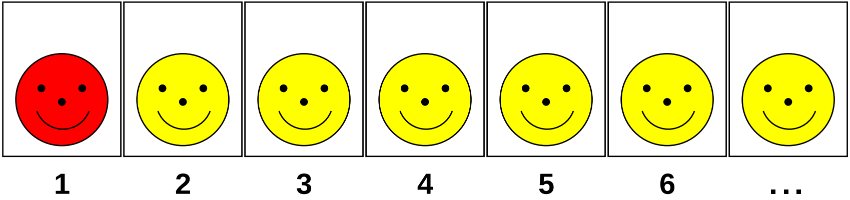

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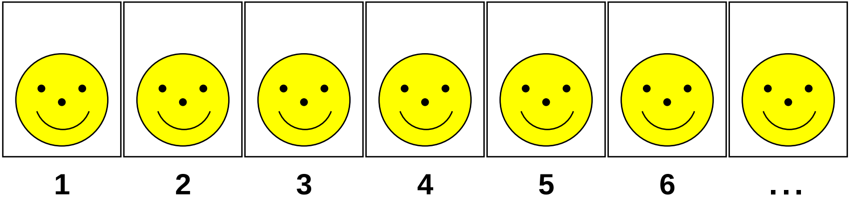

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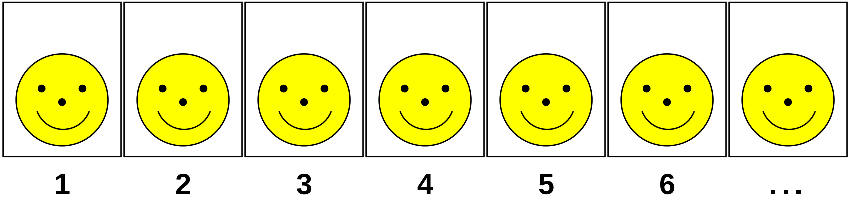

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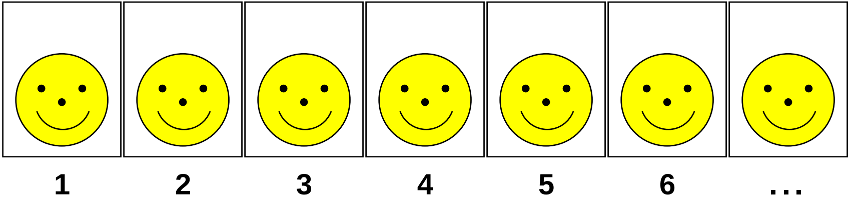

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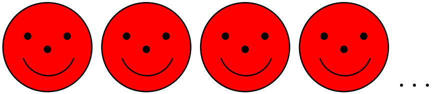

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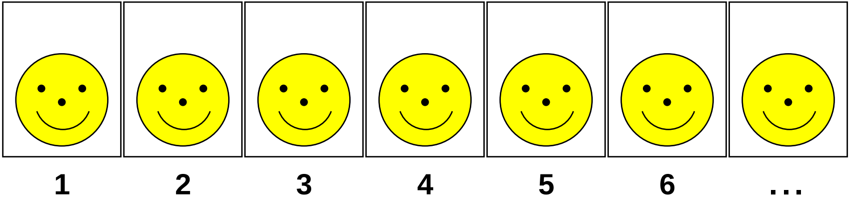

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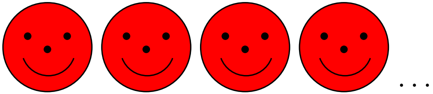

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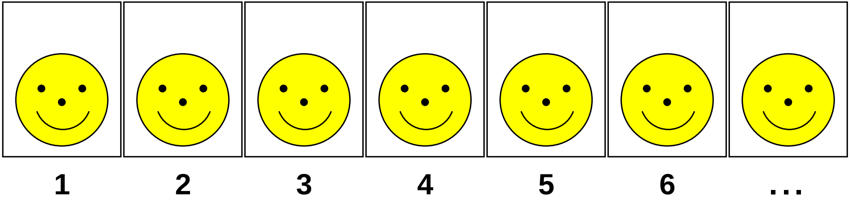

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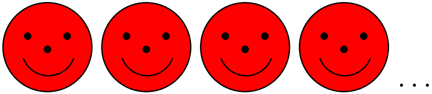

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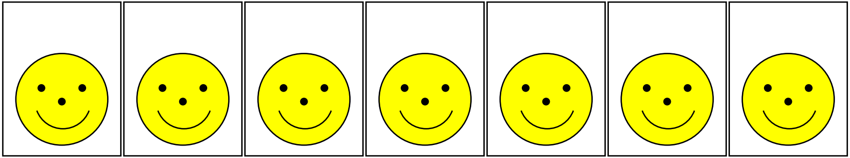

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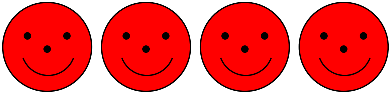

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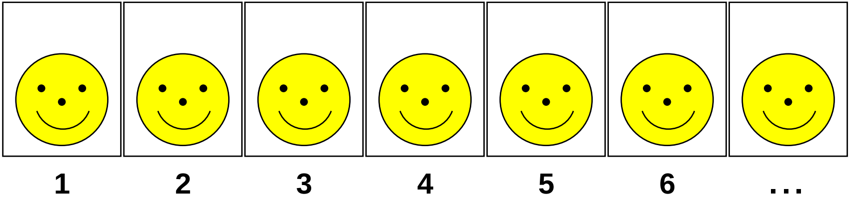

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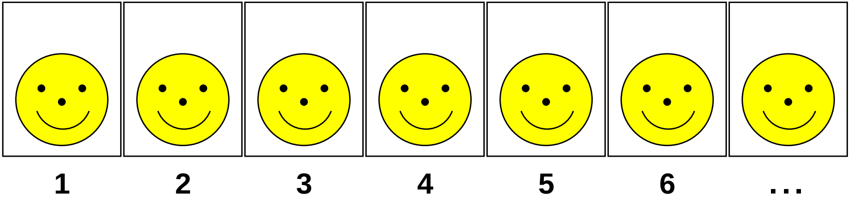

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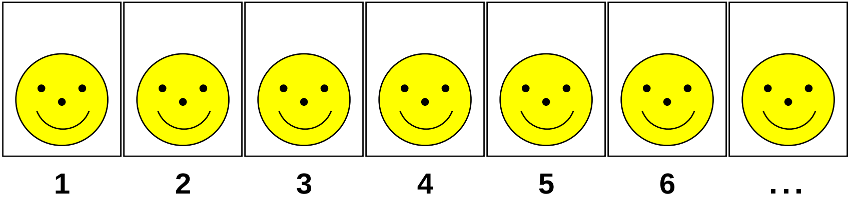

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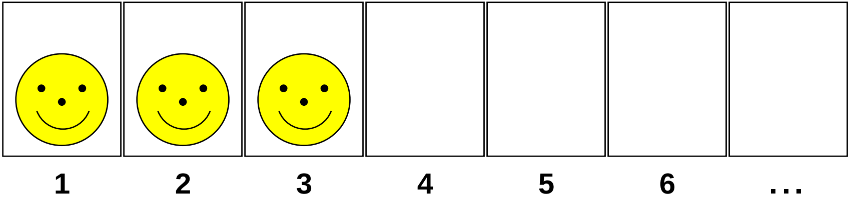

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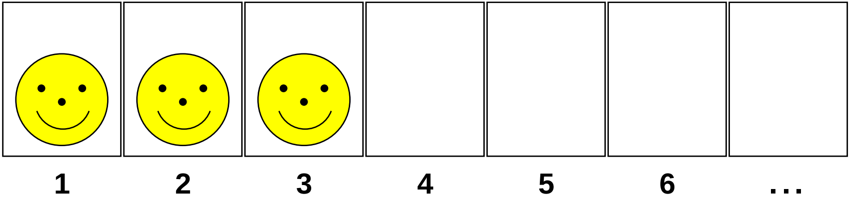

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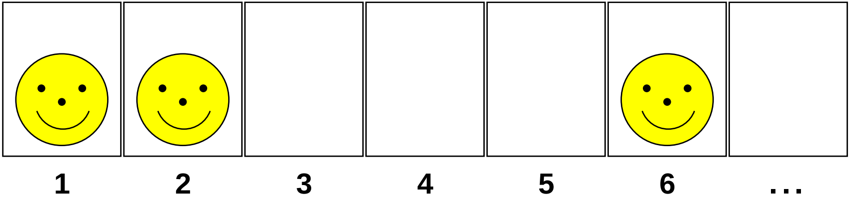

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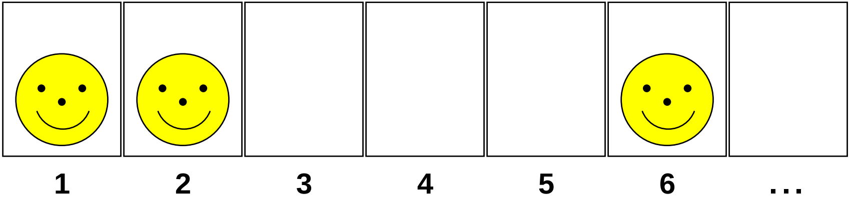

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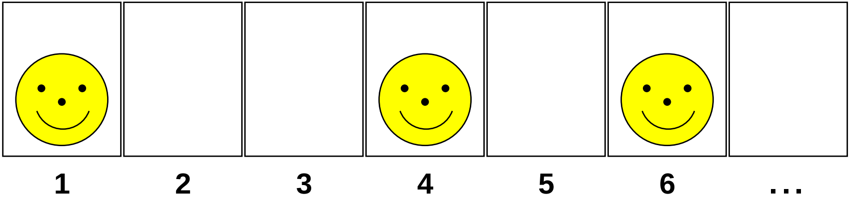

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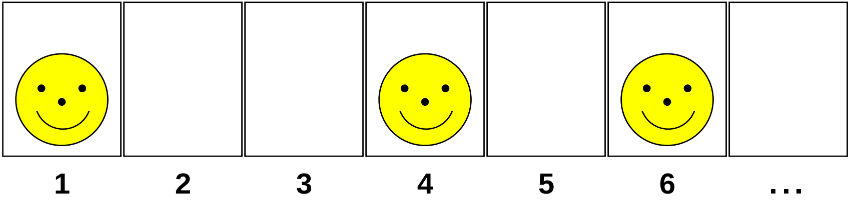

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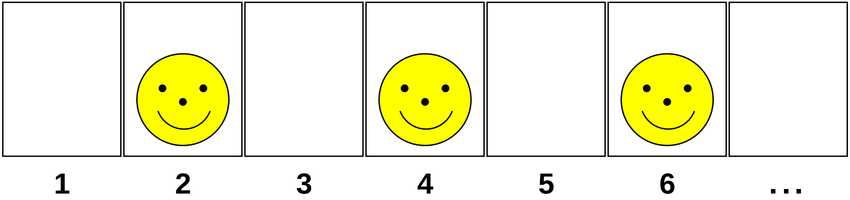

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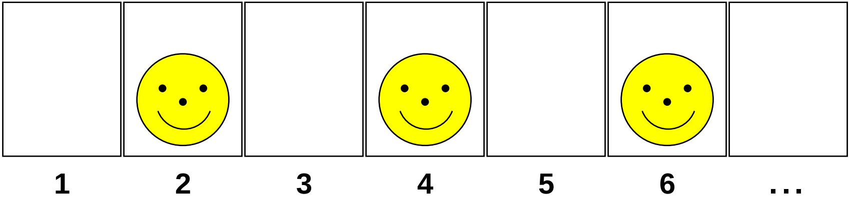

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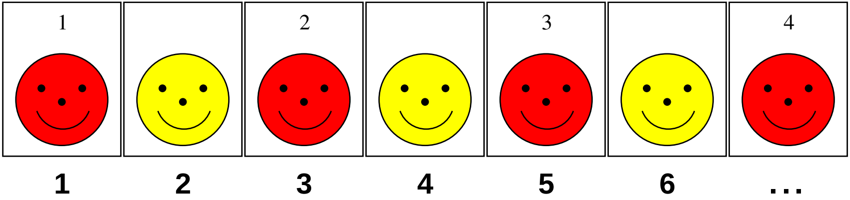

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At a stroke, Dave had not only managed to fit in all his new guests, he had even managed to free up infinitely many rooms, with minimal inconvenience to his guests!

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The next week, though, Dave finally had to admit defeat.

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So b_n is 3 or 4, and $b_n \neq a_n$.

In particular, if a guest from Cannes Tours is in room n, then their decimal expansion does not have b_n in the nth place.

Cantor's diagonalization argument (conclusion)

From previous slide

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Guest x must be in one of the rooms, say room n.
But then guest x from Cannes Tours is in room n, and has a decimal expansion with b_n in the nth place, which contradicts our choice of b_n above.

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but that's another story!

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