Banach function algebras with dense invertible group

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- In an earlier paper, Dawson and Feinstein asked whether or not a Banach function algebra with dense invertible group can have a proper Shilov boundary.
- We give an example of a uniform algebra showing that this can happen, and investigate the properties of such algebras.
- We make some remarks on the topological stable rank of commutative, unital Banach algebras.
- In particular, for an approximately regular, commutative, unital Banach algebra with character space X, we prove that the topological stable rank of A is no less than that of $C(X)$.

Introduction

Definition

Let A be a commutative, unital Banach algebra. The character space of A is denoted by Φ_A. We say that Φ_A contains analytic structure if there is a continuous injection τ from the open unit disk $\mathbb{D}$ to Φ_A such that, for all $f \in A$, $f \circ \tau$ is analytic on $\mathbb{D}$.

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In 1963, Stolzenberg gave a counter-example to the conjecture that, whenever a uniform algebra has proper Shilov boundary, its character space must contain analytic structure.

In 1968 (in his thesis) Cole gave an even more extreme example, where the Shilov boundary is proper and yet every Gleason part is trivial.

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- This leads to a new conjecture, first raised (as an open question) by Dawson and Feinstein in 2003.

New conjecture (shown to be false below)

No uniform algebra with dense invertible group can have a proper Shilov boundary.

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- Our example is of the form $P(X)$ for a compact set $X \subseteq \mathbb{C}^2$.
- In our example, the Shilov boundary is proper, and yet there is a dense set of functions in the algebra $P(X)$ whose spectra have empty interiors.

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- It is clear that this latter condition is sufficient for the invertible group to be dense in the algebra.

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- It is clear that this latter condition is sufficient for the invertible group to be dense in the algebra.
- In the final section we give a new result concerning the topological stable rank (to be defined below) of approximately regular commutative, unital Banach algebras.
- Note that a commutative, unital Banach algebra has dense invertibles if and only if it has topological stable rank equal to 1.

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The algebra C(X)

Now let X be a compact space. The algebra of continuous, complex-valued functions on X is denoted by $C(X)$.

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Let A be a unital algebra. Then $\mathrm{Inv}\, A$ denotes the invertible group of A. Now suppose that A is a unital Banach algebra. Then we say that A **has dense invertibles** if $\mathrm{Inv}\, A$ is dense in A.

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- We set $\exp A = \{\exp a : a \in A\}$.
- In the case where A is commutative, $\exp A$ is exactly the component of $\mathrm{Inv}\, A$ containing the identity.

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Definition

Let X be a compact space. A **Banach function algebra on** X is a unital subalgebra of $C(X)$ that separates the points of X and is a Banach algebra for a norm $\|\cdot\|$. Such an algebra is a **uniform algebra** if it is closed in $(C(X), |\cdot|_X)$.

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- For $f \in C(X)$, the **zero set** is $Z_X(f) = \{x \in X : f(x) = 0\}$.

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Regularity

The algebra A is **regular** if, for each proper, closed subset E of Φ_A and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $\psi(a) = 0 \ (\psi \in E)$.

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Approximate regularity

The algebra A is **approximately regular** if, for each proper, closed subset E of Φ_A and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $|\psi(a)| < 1 \ (\psi \in E)$.

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Let A be a commutative, unital Banach algebra and let E be a closed subset of Φ_A. Then A_E is the closure in $(C(E), |\cdot|_E)$ of $\{\widehat{a} \mid E : a \in A\}$.

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For example, let X be a compact subset of $\mathbb{C}$ with empty interior. Then $R(X)$ is always approximately regular but need not be regular.

The new counter-example

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It is standard that $\Phi_{P(X)}$ and $\Phi_{R(X)}$ can be identified with $\widehat{X}$ and $h_r(X)$, respectively.

With this identification, $P(\widehat{X})$ and $R(h_r(X))$ are equal to the sets of Gel'fand transforms of the elements of the algebras $P(X)$ and $R(X)$, respectively.

Main example

Theorem (Main example)

There exists a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ *in* $\mathbb{C}^2$ *such that* $(0,0) \in \widehat{Y}$, *and yet* $P(Y)$ *has dense invertibles. In particular, setting* $X = \widehat{Y}$, *the uniform algebra* $P(X)$ *is natural on* X *and has dense invertibles, but* $\Gamma_{P(X)}$ *is a proper subset of* X.

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- We sketch the construction. Let $\mathcal{F}$ be the set of all non-constant polynomials p in two variables with coefficients in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$ such that $p(\overline{\mathbb{D}}^2) \subseteq \overline{\mathbb{D}}$.

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- We construct a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ such that $(0,0) \in \widehat{Y}$ and such that, for all $p \in \mathcal{F}$, the spectrum of $p|Y$ with respect to $P(Y)$ has empty interior.

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- As emptiness of the interior of the spectrum passes to scalar multiples, it quickly follows from this that Y has the desired properties.

Sketch of construction (continued)

- As in the original example of Stolzenberg, the set Y is obtained using the (sequential) compactness, with respect to the Hausdorff metric, of the set of non-empty, closed subsets of $\overline{\mathbb{D}}^2$.

Sketch of construction (continued)

- As in the original example of Stolzenberg, the set Y is obtained using the (sequential) compactness, with respect to the Hausdorff metric, of the set of non-empty, closed subsets of $\overline{\mathbb{D}}^2$.
- Choose a countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$ which does not meet the countable set $\{p(0,0) : p \in \mathcal{F}\}$.

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- Define sets $E_{i,p}$ for $i \in \mathbb{N}$ and $p \in \mathcal{F}$ by

$$E_{i,p} = \{\underline{z} \in \overline{\mathbb{D}}^2 : p(\underline{z}) = \zeta_i\}\,.$$

  Each $E_{i,p}$ is compact, and there are only countably many such sets.

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$$E_{i,p} = \{\underline{z} \in \overline{\mathbb{D}}^2 : p(\underline{z}) = \zeta_i\}.$$

  Each $E_{i,p}$ is compact, and there are only countably many such sets.
- Enumerate those pairs $(i,p) \in \mathbb{N} \times \mathcal{F}$ for which $E_{i,p}$ is non-empty as $(i_j, p_j)_{j=1}^{\infty}$, and set $K_j = E_{i_j,p_j}$ $(j \in \mathbb{N})$.

Sketch of construction (continued)

- Adapting Stolzenberg's original technique (details available on request!) we construct inductively a sequence of successively more complicated analytic varieties W_n through $(0,0)$ (each of which is a level set of a non-constant analytic entire function on $\mathbb{C}^2$) and some closed sets $M_j \subseteq \overline{\mathbb{D}}^2$ such that $\widehat{M_j} \cap K_j = \emptyset$, and $W_n \cap \overline{\mathbb{D}}^2 \subseteq M_j$ whenever $n \geq j$.

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We now use the sequential compactness (with respect to the Hausdorff metric) of the set of non-empty, closed subsets of $\overline{\mathbb{D}}^2$.

Sketch of construction (concluded)

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- Indeed, for $p \in \mathcal{F}$, $p(\widehat{V})$ does not meet the countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$, and it is not hard to show that $\widehat{Y} = \widehat{V} \neq Y$.

Sketch of construction (concluded)

- Using the Hausdorff metric, V_n has a subsequence which converges to some non-empty, closed subset V of $\overline{\mathbb{D}}^2$.
- Clearly $(0,0) \in V$.
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- Indeed, for $p \in \mathcal{F}$, $p(\widehat{V})$ does not meet the countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$, and it is not hard to show that $\widehat{Y} = \widehat{V} \neq Y$.
- The result follows. $\square$

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Let X and Y be as constructed in our main example. It follows from our results that we have $h_r(Y) = \widehat{Y}$ and $P(Y) = R(Y)$.

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Proof. Taking the main example above as our base algebra, we may form (as in Cole's thesis) a system of root extensions to obtain a new uniform algebra A on a compact metric space such that $\{f^2 : f \in A\}$ is dense in A, and hence every point of Φ_A is a one-point Gleason part.

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Since the base algebra has proper Shilov boundary, the same is true for A (as shown by Cole). Finally, since the base algebra has dense invertibles, so does A (as shown by Dawson and Feinstein). □

Topological stable rank

We now discuss the topological stable rank and some related ranks of a commutative Banach algebra. Indeed, there is a variety of different types of stable rank discussed in the literature (see, for example, papers of Badea, Corach and Suárez, and Rieffel).

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Thus $a \in U_n(A)$ if and only if $0 \notin \sigma(a)$ if and only if $\widehat{a_1}, \ldots, \widehat{a_n}$ have no common zero on Φ_A.

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We see immediately that a commutative, unital Banach algebra A has dense invertibles if and only if $\mathrm{tsr}(A) \leq 1$.

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In particular, $C(X)$ has dense invertibles if and only if $\dim X \in \{0, 1\}$.

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This inequality is strict in many cases.

For example, if A is the disk algebra, then $\mathrm{sr}(A) = 1$, whereas $\mathrm{sr}(C(\overline{\mathbb{D}})) = 2$.

Further results

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We have strengthened the last part of this result by proving that the result holds for approximately regular, rather than regular, algebras.

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Theorem

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Let A be an approximately regular, commutative, unital Banach algebra with dense invertibles. Then $C(\Phi_A)$ *has dense invertibles.*

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Our proof is based on the standard Arens–Royden theorem, combined with some elementary topological arguments.

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We conclude with some open questions.

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- More generally (as asked by Corach and Suárez), is it always true that $\mathrm{tsr}(A) \leq \mathrm{tsr}(C(X))$, and hence $\mathrm{tsr}(C(\Phi_A)) = \mathrm{tsr}(A)$ in the case where A is approximately regular?

Warning!

We offer the following caution. Assume that one can prove that A has dense invertibles whenever A is approximately regular and $C(\Phi_A)$ has dense invertibles. Then we would have a solution to the famous 'Gel'fand problem': Is there a natural uniform algebra A on $\mathbb{I}$ such that $A \neq C(\mathbb{I})$?

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Indeed, suppose that A is a natural uniform algebra on $\mathbb{I}$. By a result of Wilken, A is approximately regular.

By our assumed result above, A has dense invertibles, and so $A = C(\mathbb{I})$ by a result of Dawson and Feinstein.

Further questions

Question 3

Let A be a uniform algebra with dense invertibles. Does it follow that $C(\Phi_A)$ has dense invertibles? More generally (as asked by Corach and Suárez), must we have $\operatorname{tsr}(C(\Phi_A)) \leq \operatorname{tsr}(A)$?

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(b) Let K be a compact subset of $\mathbb{C}^n$ such that $\widehat{K} \neq K$. Does $\widehat{K} \setminus K$ contain a homeomorphic copy of $\overline{\mathbb{D}}$?