# Banach function algebras with dense invertible group

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# Abstract


- This paper is joint work with H.G. Dales, and is available online from **http://www.maths.nott.ac.uk/∼jff/Papers/**
- In an earlier paper, Dawson and Feinstein asked whether or not a Banach function algebra with dense invertible group can have a proper Shilov boundary.
- We give an example of a uniform algebra showing that this can happen, and investigate the properties of such algebras.
- We make some remarks on the topological stable rank of commutative, unital Banach algebras.
- In particular, for an approximately regular, commutative, unital Banach algebra with character space $X$, we prove that the topological stable rank of $A$ is no less than that of $C(X)$.

# Introduction

**Definition**

Let $A$ be a commutative, unital Banach algebra. The character space of $A$ is denoted by $\Phi_A$. We say that $\Phi_A$ contains analytic structure if there is a continuous injection $\tau$ from the open unit disk $\mathbb{D}$ to $\Phi_A$ such that, for all $f \in A$, $f \circ \tau$ is analytic on $\mathbb{D}$.

In 1963, Stolzenberg gave a counter-example to the conjecture that, whenever a uniform algebra has proper Shilov boundary, its character space must contain analytic structure.

In 1968 (in his thesis) Cole gave an even more extreme example, where the Shilov boundary is proper and yet every Gleason part is trivial.

# Introduction continued

- It is elementary to show that the invertible group of $A$ cannot be dense in $A$ whenever $\Phi_A$ contains analytic structure.
- The converse is false, as is shown by the uniform algebra $C(\overline{\mathbb{D}})$ of all continuous, complex-valued functions on $\overline{\mathbb{D}}$.
- Thus the requirement that the invertible group be dense is strictly stronger than the non-existence of analytic structure in the character space of the algebra.
- This leads to a new conjecture, first raised (as an open question) by Dawson and Feinstein in 2003.

**New conjecture (shown to be false below)**

No uniform algebra with dense invertible group can have a proper Shilov boundary.

# Introduction continued

- In the next section we modify the example of Stolzenberg to show that this new conjecture is also false.
- Our example is of the form $P(X)$ for a compact set $X \subseteq \mathbb{C}^2$.
- In our example, the Shilov boundary is proper, and yet there is a dense set of functions in the algebra $P(X)$ whose spectra have empty interiors.
- It is clear that this latter condition is sufficient for the invertible group to be dense in the algebra.
- In the final section we give a new result concerning the topological stable rank (to be defined below) of approximately regular commutative, unital Banach algebras.
- Note that a commutative, unital Banach algebra has dense invertibles if and only if it has topological stable rank equal to 1.

# Introduction continued

- We recall some standard notation and results.
- In our terminology, a compact space is a non-empty, compact, Hausdorff topological space.
- We denote by $\dim X$ the covering dimension of $X$.

**The uniform norm**

Let $X$ be a non-empty set. The uniform norm on $X$ is denoted by $|\cdot|_X$.

**The algebra C(X)**

Now let $X$ be a compact space. The algebra of continuous, complex-valued functions on $X$ is denoted by $C(X)$.

# Introduction continued

## Definition

Let $A$ be a unital algebra. Then $\mathrm{Inv}\,A$ denotes the invertible group of $A$. Now suppose that $A$ is a unital Banach algebra. Then we say that $A$ **has dense invertibles** if $\mathrm{Inv}\,A$ is dense in $A$.

## Notation

Let $A$ be a commutative, unital Banach algebra, and let $a \in A$.

- The **Gel'fand transform** of $a$ is $\widehat{a} \in C(\Phi_A)$.
- The **spectrum** of $a$ is $\sigma(a)$, so that $\sigma(a) = \widehat{a}(\Phi_A)$.
- We set $\exp A = \{\exp a : a \in A\}$.
- In the case where $A$ is commutative, $\exp A$ is exactly the component of $\mathrm{Inv}\,A$ containing the identity.

# Introduction continued

## Definition

Let $X$ be a compact space. A **Banach function algebra on** $X$ is a unital subalgebra of $C(X)$ that separates the points of $X$ and is a Banach algebra for a norm $\|\cdot\|$. Such an algebra is a **uniform algebra** if it is closed in $(C(X), |\cdot|_X)$.

## Definition

Let $A$ be a Banach function algebra on a compact space $X$.

- As usual, we identify $X$ with the subset of $\Phi_A$ consisting of the evaluations at points of $X$. We say that $A$ is **natural** if $\Phi_A = X$.
- The **Shilov boundary** of $A$, denoted by $\Gamma_A$, is the minimum (non-empty) closed subset $K$ of $\Phi_A$ such that $|f|_K = |f|_{\Phi_A}$ $(f \in A)$.
- For $f \in C(X)$, the **zero set** is $Z_X(f) = \{x \in X : f(x) = 0\}$.

# Introduction continued

Let $A$ be a commutative, unital Banach algebra.

## Regularity

The algebra $A$ is **regular** if, for each proper, closed subset $E$ of $\Phi_A$ and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $\psi(a) = 0 \ (\psi \in E)$.

## Approximate regularity

The algebra $A$ is **approximately regular** if, for each proper, closed subset $E$ of $\Phi_A$ and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $|\psi(a)| < 1 \ (\psi \in E)$.

# Introduction concluded

**Notation**

Let $A$ be a commutative, unital Banach algebra and let $E$ be a closed subset of $\Phi_A$. Then $A_E$ is the closure in $(C(E), |\cdot|_E)$ of $\{\widehat{a} \mid E : a \in A\}$.

We see that $A$ is approximately regular if and only if each such uniform algebra $A_E$ is natural on $E$.

There are many uniform algebras that are approximately regular, but not regular

For example, let $X$ be a compact subset of $\mathbb{C}$ with empty interior. Then $R(X)$ is always approximately regular but need not be regular.

# The new counter-example

## Notation

- The topological boundary of a set $X \subseteq \mathbb{C}^n$ is denoted by $\partial X$.
- We shall use the notation $\underline{z} = (z, w)$ for a typical element of $\mathbb{C}^2$
- The **polynomially convex hull** and the **rationally convex hull** of $X$ are denoted by $\widehat{X}$ and $h_r(X)$, respectively.
- The algebras $P(X)$ and $R(X)$ are the uniform closures in $C(X)$ of the set of restrictions to $X$ of the polynomials and of the rational functions with poles off $X$, respectively.

It is standard that $\Phi_{P(X)}$ and $\Phi_{R(X)}$ can be identified with $\widehat{X}$ and $h_r(X)$, respectively.

With this identification, $P(\widehat{X})$ and $R(h_r(X))$ are equal to the sets of Gel'fand transforms of the elements of the algebras $P(X)$ and $R(X)$, respectively.

# Main example

**Theorem (Main example)**

*There exists a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ in $\mathbb{C}^2$ such that $(0,0) \in \widehat{Y}$, and yet $P(Y)$ has dense invertibles. In particular, setting $X = \widehat{Y}$, the uniform algebra $P(X)$ is natural on $X$ and has dense invertibles, but $\Gamma_{P(X)}$ is a proper subset of $X$.*

- We sketch the construction. Let $\mathcal{F}$ be the set of all non-constant polynomials $p$ in two variables with coefficients in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$ such that $p(\overline{\mathbb{D}}^2) \subseteq \overline{\mathbb{D}}$.
- We construct a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ such that $(0,0) \in \widehat{Y}$ and such that, for all $p \in \mathcal{F}$, the spectrum of $p|Y$ with respect to $P(Y)$ has empty interior.
- As emptiness of the interior of the spectrum passes to scalar multiples, it quickly follows from this that $Y$ has the desired properties.

# Sketch of construction (continued)

- As in the original example of Stolzenberg, the set $Y$ is obtained using the (sequential) compactness, with respect to the Hausdorff metric, of the set of non-empty, closed subsets of $\overline{\mathbb{D}}^2$.
- Choose a countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$ which does not meet the countable set $\{p(0,0) : p \in \mathcal{F}\}$.
- Define sets $E_{i,p}$ for $i \in \mathbb{N}$ and $p \in \mathcal{F}$ by

$$E_{i,p} = \{\underline{z} \in \overline{\mathbb{D}}^2 : p(\underline{z}) = \zeta_i\}.$$

  Each $E_{i,p}$ is compact, and there are only countably many such sets.
- Enumerate those pairs $(i, p) \in \mathbb{N} \times \mathcal{F}$ for which $E_{i,p}$ is non-empty as $(i_j, p_j)_{j=1}^{\infty}$, and set $K_j = E_{i_j,p_j}$ $(j \in \mathbb{N})$.

# Sketch of construction (continued)

- Adapting Stolzenberg's original technique (details available on request!) we construct inductively a sequence of successively more complicated analytic varieties $W_n$ through $(0,0)$ (each of which is a level set of a non-constant analytic entire function on $\mathbb{C}^2$) and some closed sets $M_j \subseteq \overline{\mathbb{D}}^2$ such that $\widehat{M_j} \cap K_j = \emptyset$, and $W_n \cap \overline{\mathbb{D}}^2 \subseteq M_j$ whenever $n \geq j$.
- Set $V_n = W_n \cap \overline{\mathbb{D}}^2$ $(n \in \mathbb{N})$. Note that $p_j$ does not take the value $\zeta_{i_j}$ on $\widehat{V_n}$ if $n \geq j$.
- As $n$ increases, the interiors of the sets $p(\widehat{V_n})$ $(p \in \mathcal{F})$ are thus, in some sense, gradually eliminated.

We now use the sequential compactness (with respect to the Hausdorff metric) of the set of non-empty, closed subsets of $\overline{\mathbb{D}}^2$.

# Sketch of construction (concluded)

- Using the Hausdorff metric, $V_n$ has a subsequence which converges to some non-empty, closed subset $V$ of $\overline{\mathbb{D}}^2$.
- Clearly $(0,0) \in V$.
- The set $Y = V \cap \partial \overline{\mathbb{D}}^2$ then has the desired properties.
- Indeed, for $p \in \mathcal{F}$, $p(\widehat{V})$ does not meet the countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$, and it is not hard to show that $\widehat{Y} = \widehat{V} \neq Y$.
- The result follows. $\square$

# Further results

The following result is probably known, but we know of no explicit reference.

**Theorem**

*Let $A$ be a uniform algebra with compact character space $\Phi_A$, and suppose that $A$ has dense invertibles. Then every component of every (non-empty) zero set for $A$ meets $\Gamma_A$.*

Details available on request!

Let $X$ and $Y$ be as constructed in our main example. It follows from our results that we have $h_r(Y) = \widehat{Y}$ and $P(Y) = R(Y)$.

# Further results

**Theorem**

*There is a uniform algebra $A$ on a compact metric space such that every point of $\Phi_A$ is a one-point Gleason part and such that the invertible elements are dense in $A$, but $\Gamma_A \neq \Phi_A$.*

**Proof.** Taking the main example above as our base algebra, we may form (as in Cole's thesis) a system of root extensions to obtain a new uniform algebra $A$ on a compact metric space such that $\{f^2 : f \in A\}$ is dense in $A$, and hence every point of $\Phi_A$ is a one-point Gleason part.

Since the base algebra has proper Shilov boundary, the same is true for $A$ (as shown by Cole). Finally, since the base algebra has dense invertibles, so does $A$ (as shown by Dawson and Feinstein). $\square$

# Topological stable rank

We now discuss the topological stable rank and some related ranks of a commutative Banach algebra. Indeed, there is a variety of different types of stable rank discussed in the literature (see, for example, papers of Badea, Corach and Suárez, and Rieffel).

**Definition**

Let $A$ be a commutative, unital Banach algebra, and let $a_1, \ldots, a_n \in A$. Set $a = (a_1, \ldots, a_n) \in A^n$.

- The **joint spectrum** of $a$ is denoted by $\sigma(a)$, so that $\sigma(a) = \widehat{a}(\Phi_A)$.
- The element $a$ is **unimodular** if $\sum_{i=1}^{n} Aa_i = A$ .
- We denote the set of unimodular elements of $A^n$ by $U_n(A)$.

Thus $a \in U_n(A)$ if and only if $0 \notin \sigma(a)$ if and only if $\widehat{a_1}, \ldots, \widehat{a_n}$ have no common zero on $\Phi_A$.

# Stable ranks

## Definition

Let $A$ be a commutative, unital Banach algebra.
The (**Bass**) **stable rank** of $A$ is the least $n \in \mathbb{N}$ with the property that, for all $(a_1, a_2, \ldots, a_{n+1}) \in U_{n+1}(A)$, there exists $(b_1, b_2, \ldots, b_n) \in A^n$ such that

$$(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1}) \in U_n(A)\,,$$

or $\infty$ if no such $n$ exists.
The **topological stable rank** of $A$ is the least $n \in \mathbb{N}$ such that $U_n(A)$ is dense in $A^n$, or $\infty$ if no such $n$ exists.
The stable rank and the topological stable rank of $A$ are denoted by $\mathrm{sr}(A)$ and $\mathrm{tsr}(A)$, respectively.

We see immediately that a commutative, unital Banach algebra $A$ has dense invertibles if and only if $\mathrm{tsr}(A) \leq 1$.

# Standard results on stable rank from the literature

Let $A$ be a commutative, unital Banach algebra.

- For each $n \in \mathbb{N}$, we have $\mathrm{sr}(A) \leq n$ if and only if the natural map $U_n(A) \to U_n(A/I)$ induced by the natural projection $A \to A/I$ is a surjection for every closed ideal $I$ in $A$.
- We always have $\mathrm{sr}(A) \leq \mathrm{tsr}(A)$.
- This inequality is strict for the disk algebra $A$, where $\mathrm{sr}(A) = 1$ and $\mathrm{tsr}(A) = 2$
- Let $X$ be a compact space, and set $d = \dim X$. Then

$$\mathrm{sr}(C(X)) = \mathrm{tsr}(C(X)) = [d/2] + 1$$

where $[t]$ denotes the greatest integer less than or equal to $t$.

In particular, $C(X)$ has dense invertibles if and only if $\dim X \in \{0, 1\}$.

# Standard results on stable rank from the literature

**Theorem (Corach and Suárez)**

*Let $A$ be a commutative, unital Banach algebra. Then*

$$\mathrm{sr}(A) \leq \mathrm{sr}(C(\Phi_A))\,.$$

This result depends on a deep generalization of the Arens–Royden theorem given by Taylor.

This inequality is strict in many cases.

For example, if $A$ is the disk algebra, then $\mathrm{sr}(A) = 1$, whereas $\mathrm{sr}(C(\overline{\mathbb{D}})) = 2$.

# Further results

**Theorem (Corach and Suárez again!)**

*Let $A$ be a regular Banach function algebra on $X = \Phi_A$.*

- *We have $\mathrm{sr}(A) \geq \mathrm{sr}(C(X))$, and so*

$$\mathrm{sr}(A) = \mathrm{sr}(C(X)) .$$

- *It then follows from the above results that*

$$\mathrm{tsr}(A) \geq \mathrm{tsr}(C(\Phi_A)) .$$

We have strengthened the last part of this result by proving that the result holds for approximately regular, rather than regular, algebras.

# Final results

**Theorem**

*Let $A$ be an approximately regular, commutative, unital Banach algebra. Then* $\mathrm{tsr}(A) \geq \mathrm{tsr}(C(\Phi_A))$.

Our proof is based on the standard Arens–Royden theorem, combined with some elementary topological arguments.

**Corollary**

*Let $A$ be an approximately regular, commutative, unital Banach algebra with dense invertibles. Then $C(\Phi_A)$ has dense invertibles.*

We conclude with some open questions.

# Open questions

## Question 1

Let $A$ be a uniform algebra with the property that $\exp A$ is dense in $A$.

- Can the Shilov boundary of $A$ be proper?
- Must $A$ be approximately regular?
- Must $A$ be $C(X)$?

## Question 2

Let $A$ be a uniform algebra with character space $X$. Suppose that $C(X)$ has dense invertibles.

- Must $A$ have dense invertibles
- More generally (as asked by Corach and Suárez), is it always true that $\mathrm{tsr}(A) \leq \mathrm{tsr}(C(X))$, and hence $\mathrm{tsr}(C(\Phi_A)) = \mathrm{tsr}(A)$ in the case where $A$ is approximately regular?

# Warning!

We offer the following caution. Assume that one can prove that $A$ has dense invertibles whenever $A$ is approximately regular and $C(\Phi_A)$ has dense invertibles. Then we would have a solution to the famous 'Gel'fand problem': Is there a natural uniform algebra $A$ on $\mathbb{I}$ such that $A \neq C(\mathbb{I})$?

Indeed, suppose that $A$ is a natural uniform algebra on $\mathbb{I}$. By a result of Wilken, $A$ is approximately regular.

By our assumed result above, $A$ has dense invertibles, and so $A = C(\mathbb{I})$ by a result of Dawson and Feinstein.

# Further questions

## Question 3

Let $A$ be a uniform algebra with dense invertibles. Does it follow that $C(\Phi_A)$ has dense invertibles? More generally (as asked by Corach and Suárez), must we have $\mathrm{tsr}(C(\Phi_A)) \leq \mathrm{tsr}(A)$?

## Question 4

Let $A$ be an approximately regular, commutative, unital Banach algebra. Is it true that $\mathrm{sr}(A) \geq \mathrm{sr}(C(\Phi_A))$?

## Question 5

Our final questions concern the existence of topological disks in the character space of uniform algebras.

**(a)** Let $A$ be a uniform algebra such that $\Gamma_A \neq \Phi_A$. Does $\Phi_A \setminus \Gamma_A$ contain a homeomorphic copy of $\overline{\mathbb{D}}$?

**(b)** Let $K$ be a compact subset of $\mathbb{C}^n$ such that $\widehat{K} \neq K$. Does $\widehat{K} \setminus K$ contain a homeomorphic copy of $\overline{\mathbb{D}}$?