How do we do proofs? (Part II)

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In the module G12MAN, when we talk about a sequence \((x_k)\), \textbf{by default} we assume that the sequence is infinite and that we start from \(k = 1\). In other words, the sequence has the form \(x_1, x_2, x_3, \ldots\).
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If we wish to work with sequences starting from a different first term, we will specify this. For example, a sequence of the form \(x_0, x_1, x_2, \ldots\) can be denoted by \((x_k)^\infty_{k=0}\).
Question 1

What does it mean to say that a sequence of real numbers \((x_k)\) is: Decreasing? Nondecreasing? Increasing? Strictly increasing?
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Suppose that \((x_n)\) is a sequence of real numbers. Consider the following properties that \((x_n)\) might have:

(a) \((x_n)\) is a nondecreasing sequence;
(b) \((x_n)\) is not a decreasing sequence.
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Question 2

(i) Are these two properties equivalent? (That is, does each property imply the other?)
(ii) Do either of these properties imply the other?
Different authors disagree about whether or not 0 is a natural number.

Some authors define $\mathbb{N} = \{1, 2, 3, \ldots \}$. Others define $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$. 

The answer to the next question depends on which definition of $\mathbb{N}$ you are working with: there are two possible answers for each part.

Question 3 (formal justification not required)

Let $(n_k)$ be a strictly increasing sequence of natural numbers. What is the smallest possible value that $n_2$ could have? What about $n_{200}$?

For a given positive integer $k$, what is the minimum possible value that $n_k$ could have? (Give your answer in terms of $k$.)
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**Question 4**

How would you justify your answers to Question 3 formally if you were required to?
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Note that, in the module G12MAN, 0 will not count as a natural number: for us

\[ \mathbb{N} = \{1, 2, 3, \ldots \} . \]
Question 5

Prove that, for every strictly increasing sequence of natural numbers \((n_k)\), the series \(\sum_{k=1}^{\infty} \frac{1}{n_k^2}\) is convergent.
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Question 6
Is it true that, for every strictly increasing sequence of natural numbers \((n_k)\), the series \(\sum_{k=1}^{\infty} \frac{1}{n_k}\) is convergent?
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Meta-question 5A

Can you think of several equivalent, different ways of expressing Question 5? If so, which do you find most helpful?
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**Meta-question 5A**

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**Meta-question 5B**

How could you start a formal proof here?
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Can you think of several equivalent, different ways of expressing Question 5? If so, which do you find most helpful?

Meta-question 5B

How could you start a formal proof here?

Meta-question 5C

What are you allowed to assume, state or use during the proof?
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**Meta-question 6B**

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**Meta-question 6C**

Given that you are expected to justify your answer fully, what expectations are implicit in Question 6, i.e., what do you have to do to give a satisfactory answer to Question 6?