Annotated slides and audio so far are available from my "Beamer" page. For the URL see Chapter 1: Useful Sources

3 Banach function algebras

3.1 Preliminary definitions and results

We saw in Chapter 1, via the Gelfand transform, that semisimple, unital commutative Banach algebras are (essentially) the same thing as Banach algebras of continuous functions on compact Hausdorff spaces.

Given a non-unital Banach algebra A, we know how to form the standard unitization $A^{\#}$.

Given a non-semisimple, unital, commutative Banach algebra A, the standard way to make it semisimple is to quotient out by the **Jacobson radical**.

The Jacobson radical, J, of the unital CBA A, is the intersection of all of the maximal ideals in A.

(In the non-commutative setting, you should use maximal one-sided ideals.) $\hat{A} \cong A/J$

With the **quotient norm**, A/J is then a unital, semisimple CBA. $A \xrightarrow{\wedge} \hat{A}$

From now on, we will work mostly with unital, semisimple CBA's. $\ker(\wedge) = J$

However, many of the definitions and results generalize in standard ways to all CBA's using the comments above.

For example, **most** of the named conditions we discuss hold for a given non-unital CBA A if and only if they hold for its standard unitization $A^{\#}$.

Indeed, in many cases, this can be used as the definition in the non-unital case.

Gap to fill in

A a non-unital, CBA,

then "A is regular"

$\Leftrightarrow A^{\#}/\mathrm{rad}(A^{\#})$ is regular.

Jacobson radical

In our terminology, a **compact space** is a non-empty, compact, Hausdorff topological space.

Definition 3.1.1 Let X be a compact space.

A **normed function algebra** on X is a normed algebra $(A, \|\cdot\|)$ such that A is a subalgebra of $C(X)$, such that A contains the constants and separates the points of X, and such that, for all $f \in A$, we have $\|f\| \geq |f|_X$. $\|f\|_\infty$

A normed function algebra $(A, \|\cdot\|)$ is a **Banach function algebra** on X if it is complete.

As in Chapter 1, a **uniform algebra** on X is a Banach function algebra A on X such that the norm on A is the uniform norm $|\cdot|_X$.

Of course, in the case where $(A, \|\cdot\|)$ is a Banach algebra and a subalgebra of $C(X)$, it is automatic that $\|f\| \geq |f|_X$ for all $f \in A$. (because $f \mapsto f(x)$ is a character.)

Let A be a Banach function algebra on a compact space X.

We define

or $\hat{x}$

$$\varepsilon_x : f \mapsto f(x)\,, \quad A \to \mathbb{C}\,,$$

for each $x \in X$.

Then $\varepsilon_x \in \Phi_A$, and the map $x \mapsto \varepsilon_x$, $X \to \Phi_A$, is a continuous embedding.

We say that A is **natural** (on X) if this map is surjective.

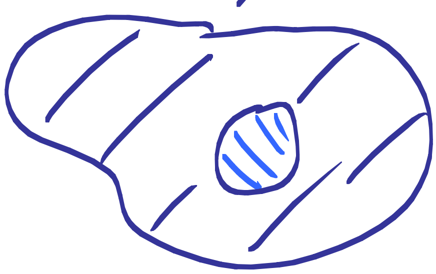
Natural if the only characters are point evals.

A typical example of a uniform algebra which is not natural is $P(X)$ when X is a compact plane set with at least one 'hole'.

$\hat{X}$ = "polynomial hull" of X:

Gap to fill in

X



$\hat{X} = X \cup$ "holes".

So add in 

Character space of $P(X)$ is naturally identified with $\hat{X}$, the polynomial hull of X.

It is standard that $C(X)$ is always natural on X.

More generally, we state a well-known result on naturality for 'full' subalgebras of $C(X)$. Here, a subalgebra A of $C(X)$ is **full** if $\operatorname{Inv} A = A \cap \operatorname{Inv} C(X)$.

Proposition 3.1.2 Let A be a Banach function algebra on a compact space X.

$f^*(x) = \overline{f(x)}$.

If A is self-adjoint, and A is a full subalgebra of $C(X)$, then A is natural on X.

Gap to fill in

<u>Proof.</u> Suppose $\phi \in \Phi_A$ and $\phi \neq \varepsilon_x$ $(x \in X)$.

(<u>Easy ex.</u>) $\forall x \in X$, $\exists\, a_x \in A$ with $a_x(x) = 1$, $\phi(a_x) = 0$.

Then $|a_x^2| = a_x a_x^* \in A$ and $\phi(\underbrace{|a_x^2|}_{>0 \text{ on some nbd of } x}) = 0$.

Add finitely many together (using compactness) to get a function $f \in \ker(\phi)$ such that $f > 0$ on all of X.

Then, since A is full, $f \in \operatorname{Inv}(A)$.

But $\phi(f) = 0$. $\Rightarrow\Leftarrow$

So no such ϕ exists, and A is natural. $\square$

Using the Gelfand transform, every unital, semisimple CBA can be regarded as a natural Banach function algebra on its character space (which is compact in this setting).

From now on, this is what we mean when we talk about **a Banach function algebra A on Φ_A.** i.e. unital case only.

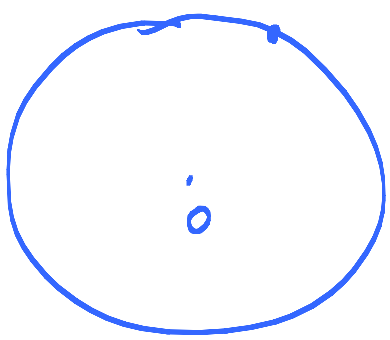
The well-known Banach algebra H^∞ (bounded analytic functions on the open unit disc) is a uniform algebra on its character space. However, this character space is quite complicated in nature.

Gap to fill in

Gelfand transform of the coordinate functional.

$\pi = \hat{Z}$

Φ_{H^∞}



Closed disc $\bar{\Delta}(0,1)$

The following useful test for naturality, due to T. Honary (1990), is not very well known.

Proposition 3.1.3 Let X be a compact space, and let $(A, \|\cdot\|)$ be a Banach function algebra on X, with uniform closure B. Then A is natural on X if and only if both of the following conditions hold:

(a) B is natural on X;

(b) $\underbrace{\lim_{n\to\infty} \|f^n\|^{1/n}}_{\sigma_A(f)} = 1$ for each $f \in A$ with $|f|_X = 1$.

The proof of this result is an **exercise**.

We can use either of the two tests for naturality above to prove that the Banach function algebra $C^1[0, 1]$ is natural on $[0, 1]$.

(There are, of course, many easy proofs of this fact.)

Gap to fill in

Easy exercise.

3.2 Two deep theorems

The next pair of theorems are very easy to state, but their proofs require the deep theory of several complex variables.

In our terminology, a **clopen** set is a set which is both open and closed.

Theorem 3.2.1 (Shilov Idempotent Theorem) Let A be a Banach function algebra on Φ_A.

Suppose that E is a clopen subset of Φ_A.

Then the characteristic function of E, χ_E, is in A.

In other words, every idempotent in $C(\Phi_A)$ is automatically in A.

Gap to fill in

Theorem 3.2.2 (Arens–Royden)

Let A be a Banach function algebra on Φ_A, and let $f \in C(\Phi_A)$.

If $\exp(f) \in A$, then $f \in A$.

In other words, whenever a function in A has a continuous logarithm defined on Φ_A, then that continuous logarithm must itself be in A.

Gap to fill in

In fact:

$$\mathrm{Inv}(A) / \exp(A) \cong \mathrm{Inv}(C(X)) / \exp(C(X))$$

$$f \exp(A) \mapsto f \exp(C(X))$$

$(f \in \mathrm{Inv}(A))$

is a group isomorphism.

3.3 Regularity and normality

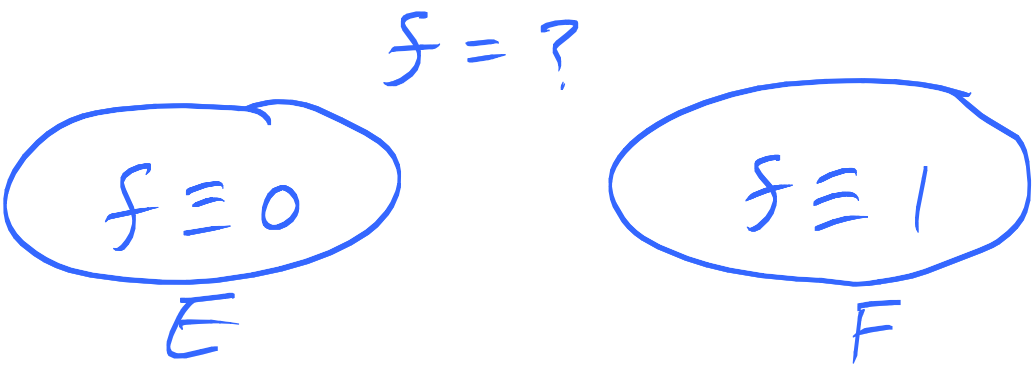
Definition 3.3.1 Let A be a Banach function algebra on a compact space X. Then A is **regular on** X if, for each proper, closed subset E of X and each $x \in X \setminus E$, there exists $f \in A$ with $f(x) = 1$ and $f(y) = 0 \ (y \in E)$;

A is **regular** if it is regular on Φ_A;

A is **normal on** X if, for each proper, closed subset E of X and each compact subset F of $X \setminus E$, there exists $f \in A$ with $f(x) = 1 \ (x \in F)$ and $f(y) = 0 \ (y \in E)$;

A is **normal** if it is normal on Φ_A.

Gap to fill in



For normality.

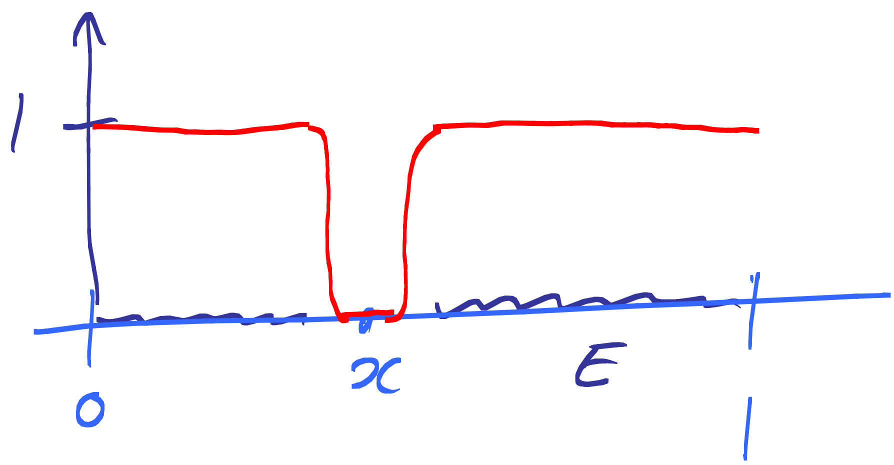
Urysohn's lemma tells us that the uniform algebra $C(X)$ is always normal, and hence also regular.

For any compact plane set X with non-empty interior, the uniform algebras $P(X)$, $R(X)$ and $A(X)$ are not regular (hence not normal). Analytic ⇒ not regular

It is easy to check that the Banach function algebra $C^1[0,1]$ is normal (and hence regular).

We will meet many more examples of regular (and normal) Banach function algebras later.

Gap to fill in



Normal uniform algebras are less common.

The first non-trivial, normal uniform algebra was constructed by McKissick in 1963.

His example was $R(X)$ for a suitable **Swiss cheese** set $X \subseteq \mathbb{C}$.

Gap to fill in

Notation: $a \in \mathbb{C}$, $r > 0$

$\Delta(a, r)$ open disc, centre a, radius r,

$\bar{\Delta}(a, r)$ = closed disc.

A Swiss cheese set is a set $X = \bar{\Delta}(a, r) \setminus \bigcup_{n=1}^{\infty} \Delta(a_n, r_n)$

$(a_n) \subseteq \mathbb{C}$, $a \in \mathbb{C}$, $r > 0$, $r_n > 0$.

[Ex. Every compact plane set is a Swiss cheese set with this defn.]

Usually add requirements such as $\sum_1^\infty r_n < \infty$,

area $(X) > 0$, int $(X) = \emptyset$

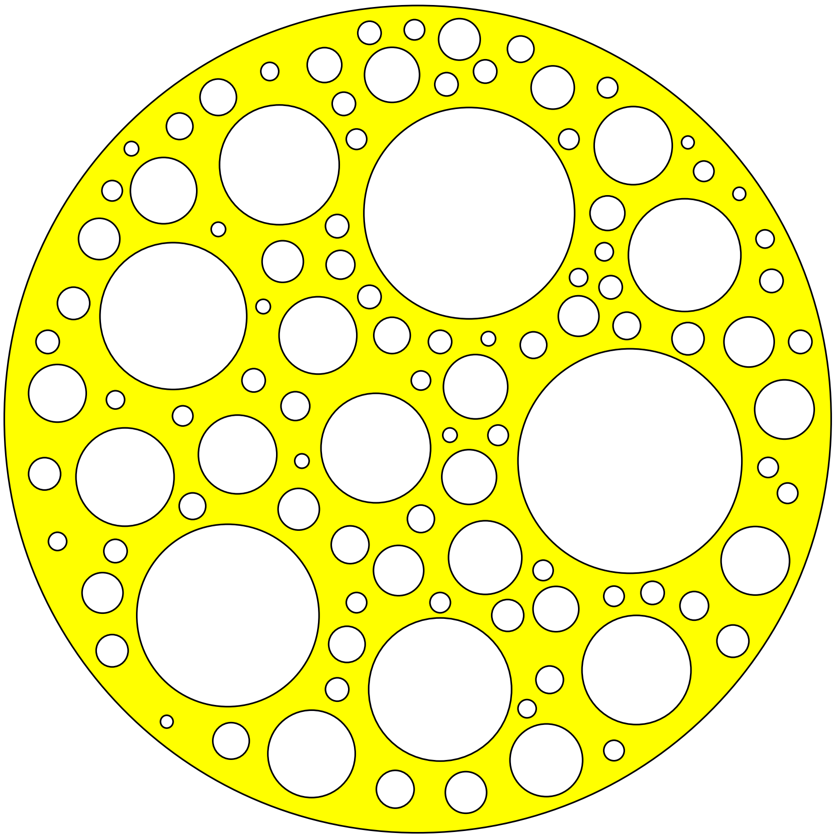
..., maybe

$\bar{\Delta}(a_n, r_n)$ are pairwise disjoint subsets of $\Delta(a, r)$.

Combining all of these requirements gives you a "classical" Swiss cheese, invented by Alice Roth (Swiss) in mid 20^{th} century.

First known examples where int $(X) = \emptyset$ but $R(X) \neq C(X)$.

McKissick's Swiss cheese was very complicated.



A Swiss cheese set with non-overlapping holes

Gap to fill in

Prob 2.1.2 Solved by Wolff in 1921. Choose a sequence of non-overlapping small discs exhausting the area of some big disc.

Say big disc is

$\bar{\Delta}(a_0, r_0)$,

smaller discs $\bar{\Delta}(a_n, r_n)$.

Insist that none of a_n are a_0.
Insist none are 0.
For any harmonic function f
on a nhd of $\bar{\Delta}(a_0, r_0)$, we

have

$$\pi r_0^2 \; f(a_0) = \int_{\bar{\Delta}(a_0,r_0)} f(w)\, \overset{\text{(area)}}{dA}(w)$$

$$= \sum_{n=1}^{\infty} \int_{\bar{\Delta}(a_n,r_n)} f(w)\, dA(w)$$

$$= \sum_{n=1}^{\infty} \pi r_n^2 \; f(a_n).$$

Let $z \in \mathbb{C}$, and set

$$f_z(w) = \exp(zw) \quad \left[\begin{array}{l}\text{entire fn}\\ \text{of } w\end{array}\right]$$

Apply above: result follows quickly.

Definition 3.3.2 Let A be a Banach function algebra on a compact space X, and let I be an ideal in A.

We define the **hull** of I, $h(I)$, to be the intersection of the zero sets of the functions in I, i.e.,

$$h(I) = \bigcap_{f \in I} f^{-1}(\{0\}) .$$

$= \{x \in X \mid \varepsilon_x(I) = \{0\}\}$

Now let E be a subset of X.

We define the **kernel** of E, $I(E)$ (sometimes denoted by $k(E)$), to be the closed ideal in A,

$$\{f \in A : f(E) \subseteq \{0\}\},$$

consisting of those functions in A which vanish identically on E.

Suppose that A is natural on X, so that we may regard X as equal to Φ_A.

In this setting, since every proper ideal is contained in a maximal ideal, it follows that **no proper ideal in A has empty hull**.

This is false without the assumption of naturality.

I, J ideals, then $h(I+J)$ $= h(I) \cap h(J)$.

Ex. $h(I(h(J))) = h(J)$.

The connection with regularity is given by the following standard result.

> **Proposition 3.3.3** Let A be a Banach function algebra on a compact space X.
>
> Then A is regular on X if and only if, for every closed subset E of X, we have $h(I(E)) = E$.

$\Leftrightarrow$ every Gelfand closed set is a hull.

More generally, we can define the **hull-kernel topology** on X using $E \mapsto h(I(E))$ as a closure operation.

Let A be a Banach function algebra on Φ_A.

Then the hull-kernel topology on Φ_A is weaker than the Gelfand topology, and the topologies agree if and only if A is regular.

Otherwise, the hull-kernel topology is non-Hausdorff.

(By above, every hull is $\underbrace{\text{hull-kernel}}_{\text{h-k}}$ closed)

In the next result, it is not enough to assume **regularity on** X, unless you know that your algebra is natural on X.

Theorem 3.3.4 Every regular Banach function algebra is normal.

Gap to fill in

Let $E, F \subseteq \Phi_A$ with $E \cap F = \phi$, E closed, F compact.

A regular, so $h(I(E)) = E$
$h(I(F)) = F$.

$h(I(E) + I(F)) = h(I(E)) \cap h(I(F))$
$= E \cap F = \phi$.

Thus $I(E) + I(F) = A$.

So $\exists\, f \in I(E),\ g \in I(F)$ with $f + g = 1$.

Then $f \equiv 0$ on E, $f \equiv 1$ on F

$g \equiv 1$ on E, $g \equiv 0$ on F.

$\square$

This argument shows that whenever A is natural, I, J ideals with $h(I) \cap h(J)$ empty, then we can find f, g as above.

In particular, if E, F are disjoint $\underbrace{h\text{-}k}_{\text{hull-kernel}}$ closed sets in Φ_A, then we can find f, g as above.

$$f \equiv 0 \text{ on } E, \quad f \equiv 1 \text{ on } F$$

$$g \equiv 1 \text{ on } E, \quad g \equiv 0 \text{ on } F$$

$$f + g = 1.$$

The definition of normality is reminiscent of Urysohn's lemma.

However, in view of the following result of Bade and Curtis, we can not hope to maintain control of the norms of the functions f separating closed sets from each other.

This result (and its proof) may be found as Theorem 4.1.19 in the book of Dales.

Theorem 3.3.5 Let $(A, \|\cdot\|)$ be a Banach function algebra on a compact space X, let $M > 0$, and let $c \in (0, 1/2)$.

Suppose that, for every pair of disjoint closed subsets E and F of X, there is an $f \in A$ with $\|f\| \leq M$ and such that $|f(x)| < c \ (x \in E)$ and $|f(x) - 1| < c \ (x \in F)$.

Then $A = C(X)$.

3.4 Some examples of Banach function algebras

We have already discussed the uniform algebras $C(X)$ (for compact spaces X) and $P(X)$, $R(X)$ and $A(X)$ (for compact plane sets X).

Obviously, we can also work with compact subsets of $\mathbb{C}^N$ for $N > 1$.

We also mentioned the Banach function algebra $C^1[0,1]$.

Similarly, we can define $C^n[0,1]$, the Banach function algebra of n-times continuously differentiable complex-valued functions on the interval $[0,1]$.

However, the algebra $C^\infty[0,1]$ of infinitely-differentiable functions on $[0,1]$ is not a Banach function algebra (no matter which norm you try).

Gap to fill in

<u>Hint:</u> commutative Singer–Wermer theorem (easy special case).

Other well-known Banach function algebras include the Lipschitz algebras $\mathrm{Lip}_{\alpha}(X)$ and $\mathrm{lip}_{\alpha}(X)$ for a compact metric space X, and Fourier algebras $A(\Gamma) = L^1(G)$ where G and Γ are mutually dual, locally compact, abelian groups.

These Banach function algebras are regular (and, indeed, have many stronger properties).

See Sections 4.4 and 4.5 of the book of Dales for many more details concerning these algebras.

Gap to fill in

There are also many useful examples of Banach function algebras on $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, the one-point compactification of $\mathbb{N}$.

Here, $C(\mathbb{N}_\infty)$ may be identified with the algebra of all convergent complex sequences.

Exercise.

(i) **(You may wish to quote the Shilov Idempotent Theorem!)** Prove that every natural Banach function algebra on $\mathbb{N}_\infty$ is regular.

(ii) Give an example of a Banach function algebra on $\mathbb{N}_\infty$ that is not regular on $\mathbb{N}_\infty$.

Gap to fill in

For $1 \leq p < \infty$, the usual sequence spaces ℓ^p, are (non-unital) subalgebras of $C(\mathbb{N}_\infty)$, and these subalgebras may be unitized to give natural, Banach function algebras on $\mathbb{N}_\infty$.

Another interesting family of Banach function algebras on $\mathbb{N}_\infty$ is the following.

Let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be a sequence of positive real numbers.

We define A_α by

$$A_\alpha = \left\{ f \in C(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)| < \infty \right\}.$$

It is easy to see that A_α is a subalgebra of $C(\mathbb{N}_\infty)$, and that A_α is a Banach function algebra, where the norm of a function $f \in A_\alpha$ is given by

$$\|f\| = \|f\|_\infty + \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)|.$$

It is also easy to check that the character space of A_α is just $\mathbb{N}_\infty$.

We will frequently return to these examples in the remaining chapters.

3.5 Regularity and decomposable operators

We conclude this chapter by discussing an interesting connection, due to M. Neumann (1992), between regularity and the theory of **decomposable operators**.

Definition 3.5.1 Let E be a Banach space, and let $T \in B(E)$. Then T is **decomposable** if, for every open cover $\{U, V\}$ of $\mathbb{C}$, there are closed, invariant subspaces F and G for T such that $E = F + G$, $\sigma(T|_F) \subseteq U$ and $\sigma(T|_G) \subseteq V$.

Gap to fill in

$U, V \subseteq \mathbb{C}$, $\quad U \cup V = \mathbb{C}$.

$U^c = \mathbb{C} \setminus U \quad , \quad V^c = \mathbb{C} \setminus V$

$\updownarrow \qquad\qquad \updownarrow$

$C_1 \qquad\qquad C_2$

$\sigma(T|_F) \cap C_1 = \emptyset, \quad \sigma(T|_G) \cap C_2 = \emptyset$

T has *Weak 2-SDP*

if [. . . . as decomposable except] $F+G$ is dense in E.

---

decomposable $\overset{\not\Leftarrow}{\Rightarrow}$ Weak 2-SDP

T is *superdecomposable* if

T is decomposable, but we can find an operator R with $RT = TR$, and can take

$$F = \overline{R(E)},$$
$$G = \overline{(I-R)E}.$$

Superdecomposable $\underset{\not\Leftarrow}{\Rightarrow}$ decomposable.

Let A be a Banach function algebra. For each $f \in A$, we denote by T_f the multiplication operator in $B(A)$ defined by $g \mapsto fg$.

Neumann investigated the decomposability of these multiplication operators, and proved the following.

Proposition 3.5.2 (M. Neumann, 1992) Let A be a Banach function algebra on Φ_A.

For each $f \in A$, the multiplication operator T_f is decomposable if and only if the function f is continuous when Φ_A is given the **hull-kernel** topology.

Thus the Banach function algebra A is regular if and only if, for all $f \in A$, the multiplication operator T_f is decomposable.

Gap to fill in

This "Corollary" was known earlier.

Colojoara and Foias 1968

A regular $\Rightarrow$ all T_f decomposable.

Frunza 1973,

All T_f decomposable $\Rightarrow$ A is regular.

In fact, Neumann proved the following. For $f \in A$

Ban fn. alg.

T.F.A.E.

(a) T_f is decomposable

(b) T_f has Weak 2-SDP

(c) T_f is superdecomposable

(d) f is cts $\Phi_A \to \mathbb{C}$ when Φ_A has h-k topology.

Fact

Let A be a Ban. fn algebra on Φ_A. Let Y be a h-k closed subset of Φ_A. Then (without h-k closed)

$$A|_Y \cong A/_{I(Y)}, \text{ so}$$

$A|_Y$ is a Ban. fn. alg on Y.

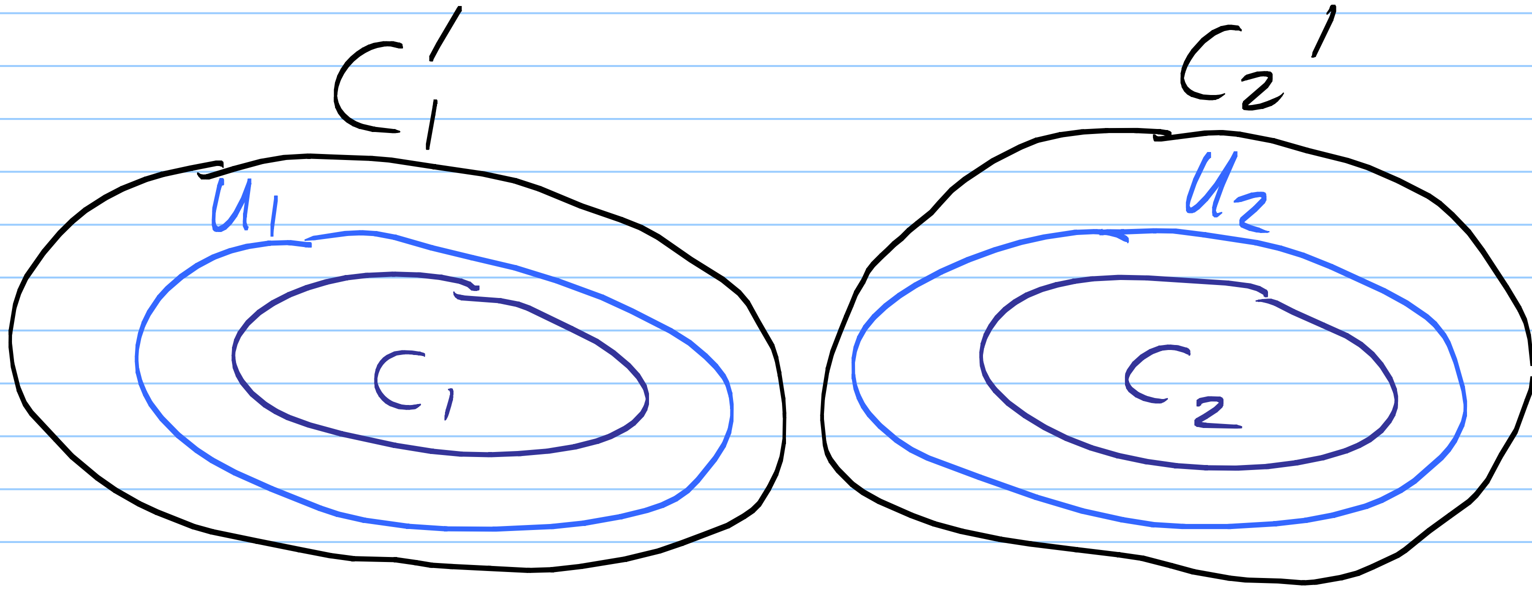
Given Y is h-k closed, then $A|_Y$ is a <u>natural</u> Ban fn. algebra on Y.

In particular, if $f \in A$ and $0 \notin f(Y)$, then $\exists$ $a \in A$ s.t. $af \equiv 1$ on Y.

<u>Proof of strong version of 3.5.2.</u>

First, suppose that f is cts when Φ_A has h-k top.

We show T_f is superdecomposable. Let C_1, C_2 be disjoint closed $\subseteq \mathbb{C}$.



Choose open sets U_1, U_2 closed sets C_1' and C_2' with

$$C_1 \subseteq U_1 \subseteq C_1', \quad C_2 \subseteq U_2 \subseteq C_2',$$

and $C_1' \cap C_2' = \phi$.

Since f is cts w.r.t. h-k top, $f^{-1}(C_1')$, $f^{-1}(C_2')$ are disjoint h-k closed sets in Φ_A.

So there is $g \in A$
with $g \equiv 1$ on $f^{-1}(C_1')$
$g \equiv 0$ on $f^{-1}(C_2')$

Show that $\sigma\left(T_f\big|_{\overline{gA}}\right) \cap C_2 = \phi$.

Let $\lambda \in C_2$.

Then we have
$\lambda \notin \mathbb{C} \setminus U_2$.
$f^{-1}(\mathbb{C} \setminus U_2)$ is h-k closed in $\bar{\Phi}_A$. Since $f-\lambda$ is never 0 on this set, by comments above, there is an $a_\lambda \in A$ with $a_\lambda(f-\lambda)\big|_{f^{-1}(\mathbb{C} \setminus U_2)} = 1$.

So $a_\lambda(f-\lambda) \equiv 1$ on supp g,
and $a_\lambda(f-\lambda)\, g = g$.

Hence $T_{a_\lambda}(T_f - \lambda I)\big|_{\overline{gA}} = Id_{\overline{gA}}$.

So $\lambda \notin \sigma(T_f|_{\overline{gA}})$.

Thus $\sigma(T_f|_{\overline{gA}}) \cap C_2 = \phi$

and, replacing g by $1-g$

same argument shows

$$\sigma(T_f|_{\overline{(1-g)A}}) \cap C_1 = \phi.$$

For "converse" (weak 2-SDP $\Rightarrow$ f dz when Φ_A has h-k top)

see Neumann's paper.

$\square$