4 Regularity conditions and their implications

4.1 Regularity and the structure of ideals

Recall the notions of the **hull** $h(I)$ of an ideal I (the intersection of the zero sets of the functions in I) and the **kernel** $I(E)$ of a set E (defined again below).

We now restrict attention to closed sets E.

Let A be a Banach function algebra on a compact space X.

For each closed set $E \subseteq X$ we define a pair of ideals, $I(E)$ (as before) and $J(E)$ by

$$I(E) = \{f \in A : f(E) \subseteq \{0\}\}$$

and

$$J(E) = \{f \in A : E \subseteq \operatorname{int}\left(f^{-1}(\{0\})\right)\}.$$

Thus the functions in $I(E)$ are 0 at all points of E, while each function $f \in J(E)$ is 0 at all points of some neighbourhood of E: the neighbourhood depends on f.

For $x \in X$, we denote the maximal ideal $I(\{x\})$ by M_x, and we write J_x for $J(\{x\})$.

Even without regularity, we always have $h(M_x) = \{x\}$.

However, an easy compactness argument shows that A is regular on X if and only if, for all $x \in X$, we have $h(J_x) = \{x\}$.

Gap to fill in

A is <u>regular on X</u>

if, for all x, y in X with $x \neq y$, $J_x \not\subseteq M_y$.

Consider disc algebra $A(\bar{\Delta})$. Then all J_x are $\{0\}$.

The next result shows a close connection between regularity and the ideal structure.

This result (and its proof) may be found as Proposition 4.1.20 in the book of Dales.

Proposition 4.1.1 Let A be a regular Banach function algebra on Φ_A, and let E be a closed subset of Φ_A. Then:

I an ideal in A and h(I) = E ⇒ J(E) ⊆ I.

(i) $J(E)$ is the minimum ideal in A whose hull is E;

(ii) $\overline{J(E)}$ is the minimum **closed** ideal in A whose hull is E;

(iii) $I(E)$ is the maximum ideal in A whose hull is E;

(iv) for each ideal I in A, we have

$$J(S) \subseteq I \subseteq I(S)\,,$$

where $S = h(I)$.

for every closed ideal I,

$\overline{J(S)} \subseteq I \subseteq I(S)$

(again $S = h(I)$).

In this setting, the quotient algebras $I(E)/\overline{J(E)}$ provide a good source of **radical Banach algebras**.

However, this topic is beyond the scope of this course.

Recall that a **prime ideal** in a commutative, complex algebra A is a **proper** ideal P in A with the property that $A \setminus P$ is multiplicatively closed.

Note that every maximal ideal in a unital CBA is a closed prime ideal, and that the trivial ideal $\{0\}$ is prime if and only if the algebra is an integral domain.

Gap to fill in

If $ab \in P$ then $a \in P$ or $b \in P$.

The next standard result places restrictions on the possible prime ideals in regular algebras.

Proposition 4.1.2 Let A be a regular Banach function algebra on Φ_A, and let P be a prime ideal in A. Then $h(P)$ has only one element, x say, and we have

$$J_x \subseteq P \subseteq M_x \,.$$

Gap to fill in

Easy consequence of regularity.

If P is a <u>closed</u> prime ideal then $\overline{J_x} \subseteq P \subseteq M_x$.

4.2 Spectral synthesis and strong regularity

Definition 4.2.1 Let A be a Banach function algebra on a compact space X, and let E be a closed subset of X.

We say that E is a **set of synthesis** for A if
$\overline{J(E)} = I(E)$.

Now let $x \in X$. We say that A is **strongly regular at** x if $\{x\}$ is a set of synthesis for A, i.e., if $\overline{J_x} = M_x$.

The Banach function algebra is **strongly regular** on X, if it is strongly regular at all points of X.

Spectral synthesis holds for A (or A **has spectral synthesis**) on X if every closed subset of X is a set of synthesis for A.

Clearly, whenever spectral synthesis holds for A on X, then A is strongly regular on X.

If $J_x \subseteq M_y$ then $\overline{J_x} \subseteq M_y$. So if $\overline{J_x} = M_x$, this only happens for $x = y$.

It is also easy to see that, if A is strongly regular on X, then A is regular on X.

However, an elegant argument of Mortini shows that, whenever A is strongly regular on X, then A is natural on X.

Thus every such algebra is natural and regular (and hence normal), and we may, without ambiguity, omit the 'on X' above.

Gap to fill in

Exercise!

4.3 Examples

At this point, let us see where our previous examples fit in.

- For every compact space X, $C(X)$ has spectral synthesis, and is strongly regular (etc.).
- For each $n \in \mathbb{N}$, the Banach function algebra $C^n[0,1]$ is **not** strongly regular.

Gap to fill in

In $C^1[0,1]$,

$$\overline{J_x} = \{ f \in C^1[0,1] \mid f(x) = f'(x) = 0 \}$$

$$\neq M_x .$$

- (Sherbert, 1964) For every compact metric space X and $\alpha \in (0,1)$, the 'little' Lipschitz algebras $\mathrm{lip}_\alpha(X)$ have spectral synthesis.

  However, for $\alpha \in (0,1]$, the **only** closed sets which are of synthesis for $\mathrm{Lip}_\alpha(X)$ are the **clopen** subsets of X.

  Thus, unless the compact space X is discrete (and hence finite), $\mathrm{Lip}_\alpha(X)$ is not strongly regular, and does not have spectral synthesis.

- Let Γ be a locally compact, abelian group. Then the Fourier algebra $A(\Gamma)$ is strongly regular.

  However, it is a famous theorem of Malliavin that $A(\Gamma)$ does not have spectral synthesis unless Γ is discrete.

Gap to fill in

- All of the examples of Banach sequence algebras discussed earlier have spectral synthesis, and hence are strongly regular.

  In general, regular Banach sequence algebras need not be strongly regular.

- Let X be a compact plane set such that $R(X) \neq C(X)$.

  It is not known whether $R(X)$ can be strongly regular, or whether $R(X)$ can have spectral synthesis.

- It is open whether or not a non-trivial uniform algebra can have spectral synthesis.

- There are non-trivial, strongly regular uniform algebras. The first known examples were due to Feinstein (1992).

Gap to fill in

Start with $R(X)$ regular and $\neq C(X)$ (McKissick's cheese).

Use Cole's method to

adjoin square roots until you end up with a uniform algebra A on a compact metric space $X = \Phi_A$ such that $\{f^2 \mid f \in A\}$ is dense in A:

in fact A is then also normal. This then implies A is strongly regular (and more).

4.4 Spectral synthesis and closed ideals

Let A be a Banach function algebra on a compact space X.

Suppose that A is strongly regular. Then A is natural on X, A is regular, and, for all $x \in X$, the maximal ideal M_x is the unique closed ideal in A whose hull is $\{x\}$.

In particular, every closed prime ideal in A is a maximal ideal.

Now suppose, further, that A has spectral synthesis.

In this case, we have a complete description of the closed ideals in A: they are precisely the kernels of the closed subsets of A.

Gap to fill in

Let I be a closed ideal, set $S = h(I)$. Then

$I(S) = \overline{J(S)} \subseteq I \subseteq I(S)$

so $I = I(S)$.

Conversely, a Banach function algebra A on Φ_A has spectral synthesis if and only if A is regular, and the only closed ideals in A are the kernels $I(E)$ of closed sets $E \subseteq \Phi_A$.

It appears to be open whether or not the regularity assumption here is redundant:

Suppose that the only closed ideals in A are the kernels $I(E)$ of closed sets $E \subseteq \Phi_A$.

Does it follow that A is regular, and hence that A has spectral synthesis?

Gap to fill in

4.5 Bounded approximate identities and strong Ditkin algebras

We begin this section by recalling the definition of bounded approximate identity.

Definition 4.5.1 Let B be a commutative Banach algebra (usually **without** identity).

A **bounded approximate identity** in B is a bounded net $(e_\alpha) \subseteq B$ such that, for all $b \in B$, the net $(e_\alpha b)$ converges to b in B.

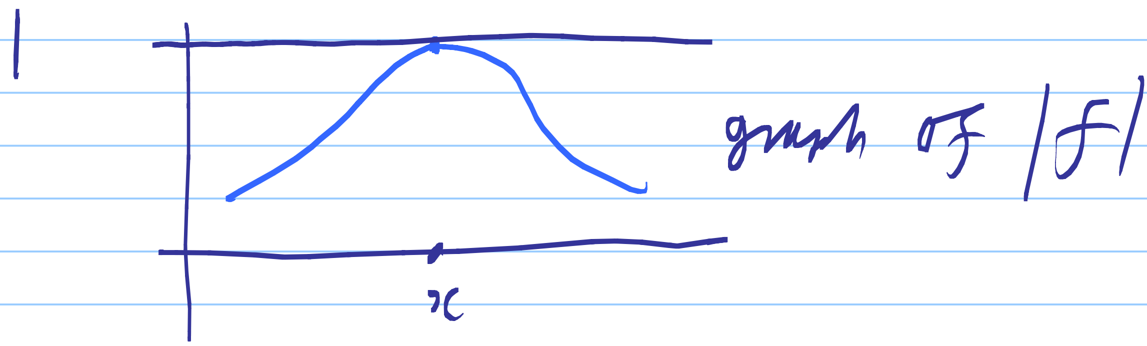
Gap to fill in

Uniform algebras:

Let A be a uniform algebra on a compact metric space X. Then for x ∈ X, M_x has a b.a.i.

⟺ x is a "peak point" for A

i.e. there is a function $f \in A$ with $f(x) = 1$ and $|f(y)| < 1$ $(y \in X \setminus \{x\})$.



[Can take a sequential b.a.i. given by $e_n = 1 - f^n$.]

We now introduce the strong Ditkin algebras.

Definition 4.5.2 Let A be a Banach function algebra. Then A is a **strong Ditkin algebra** if A is strongly regular, and every maximal ideal in A has a bounded approximate identity.

There is an intermediate condition between being **strongly regular** and being a **strong Ditkin algebra**, namely being a 'Ditkin algebra'.

We shall not discuss this condition in detail here, but those interested may wish to consult Chapter 4 of the book of Dales.

Gap to fill in

From now on we use the standard abbreviation **b.a.i.** for bounded approximate identity.

Returning to our standard examples, we have the following.

- For every compact space X, $C(X)$ is a strong Ditkin algebra.

- Consider the algebra $\mathrm{lip}_{\alpha}(X)$, where X is a compact metric space and $\alpha \in (0, 1)$.

  Let $x \in X$. Then M_x has a b.a.i. if and only if x is an isolated point of X.

  Thus, if X is infinite, then $\mathrm{lip}_{\alpha}(X)$ is a Banach function algebra which has spectral synthesis, but is **not** a strong Ditkin algebra.

- For every locally compact abelian group Γ, the Fourier algebra $A(\Gamma)$ **is** a strong Ditkin algebra.

  Thus, if Γ is not discrete, then $A(\Gamma)$ is a strong Ditkin algebra which does not have spectral synthesis.

- For $p \in [1, \infty)$, the (standard unitizations of the) Banach sequence algebras ℓ_p are not strong Ditkin algebras.

Gap to fill in

- The situation for the algebras A_α depends on the sequence α.

  Recall that

$$A_\alpha = \left\{ f \in C(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} \alpha_n |f(n+1) - f(n)| < \infty \right\}.$$

  where $\alpha = (\alpha_n)$ is a sequence of positive real numbers.

  In this case, A_α is a strong Ditkin algebra if and only if (α_n) has a bounded subsequence.

  Thus A_α fails to be a strong Ditkin algebra if and only if $\alpha_n \to +\infty$ as $n \to \infty$.

Gap to fill in

Look at $f_n(k) = \begin{cases} 1 & \text{for } n \le k \\ 0 & \text{for } n > k. \end{cases}$

If (α_n) has bounded subsequence

then (f_n) have a bounded subseq. and this is a b.a.i. in M_∞.

(By Bishop's theorem)

For a compact plane set X, $R(X)$ is a strong Ditkin algebra if and only if $R(X) = C(X)$.

However, there are non-trivial, strong Ditkin uniform algebras. See [F. 1992]

Strongly regular uniform algebras need not be strong Ditkin algebras.

Gap to fill in

4.6 Automatic continuity and Banach extensions for strong Ditkin algebras

We now recall the definitions of Banach A-bimodules, derivations, and intertwining maps, which play an important role in the cohomology theory of Banach algebras.

Definition 4.6.1 Let A be a commutative Banach algebra.

A **Banach A-bimodule** is an A-bimodule E which is also a Banach space such that both of the bilinear maps $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto x \cdot a$ are jointly continuous from $A \times E$ to E.

Given a Banach A-bimodule E, a linear map D from A to E is a **derivation** if, for all $a, b \in A$ we have

$$D(ab) = a \cdot D(b) + D(a) \cdot b\,.$$

A linear map T from A to E is an **intertwining map** if, for all $a \in A$, both of the maps $b \mapsto T(ab) - a \cdot T(b)$ and $b \mapsto T(ba) - T(b) \cdot a$ are continuous from A to E.

Clearly, every derivation into a Banach A-bimodule is an intertwining map.

Gap to fill in

Ex.

A powerful automatic continuity result holds for strong Ditkin algebras.

This result, and its proof, may be found as Corollary 5.3.5 in the book of Dales.

Theorem 4.6.2 Let A be a strong Ditkin algebra and let E be a Banach A-bimodule.

Then every intertwining map from A to E is automatically continuous.

In particular, every derivation from a strong Ditkin algebra A into a Banach A-bimodule is automatically continuous.

Gap to fill in

See book of Dales:

separating space, continuity ideal, gliding hump theorem, stability lemma ...

Another important cohomological property of strong Ditkin algebras concerns the splitting of Banach extensions.

The following theorem, and its proof, may be found as Theorem 5.4.41 in the book of Dales.

Theorem 4.6.3 Let A be a strong Ditkin algebra or a C^*-algebra.

Then every Banach extension of A which splits also splits strongly.

Gap to fill in

Let A be a Banach algebra.
Then a Banach extension of A is a short exact sequence

$$0 \to I \overset{\iota}{\hookrightarrow} \mathfrak{A} \xrightarrow{\pi} A \to 0$$

where $\mathfrak{A}$ is a Banach algebra,

π is a continuous algebra homomorphism, and

I is a closed ideal in $\mathfrak{A}$

ι is inclusion map.

Note $I = \ker \pi$.

In fact, given a Banach algebra $\mathfrak{A}$ and a surjective alg. hom $\pi : \mathfrak{A} \to A$, we can form

$$0 \to \ker \pi \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} A \to 0$$

<u>Example</u> Let $\mathfrak{A}$ be unital CBA. Set $I = \mathrm{rad}(\mathfrak{A})$ (Jacobson radical). Set $A = \mathfrak{A}/I$. So A is then a Ban. fn. algebra

$$0 \longrightarrow \underbrace{\mathrm{rad}(\mathfrak{A})}_{I} \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} \underbrace{\mathfrak{A}/I}_{A} \to 0$$

Returning to general extension

$$0 \longrightarrow I \hookrightarrow \mathfrak{A} \underset{\theta}{\overset{\pi}{\rightleftarrows}} A \to 0$$

The Banach extension *splits* if $\exists$ alg. hom $\theta : A \to \mathfrak{A}$ with $\pi\theta = Id_A$.

The extension splits *strongly* if there is a continuous such θ.

[This connects with Wedderburn decompositions when you take $I = rad(\mathfrak{A})$ as above.]

Separating space.

Let $T : E \to F$

E, F Banach spaces, T linear.

We know (Closed graph thm)

T is cts $\iff$ T has closed graph

$\Leftrightarrow$ whenever $x_n \to 0$ in E
and $Tx_n \to y$ in F
then y must be 0.

The <u>separating space</u> of T
is $\mathfrak{S}(T)$ defined by

$$\mathfrak{S}(T) = \{ y \in F \mid \exists \ (x_n) \subseteq E \text{ with } x_n \to 0 \text{ and } Tx_n \to y \text{ as } n \to \infty \}.$$

So T is cts $\Leftrightarrow$ $\mathfrak{S}(T) = \{0\}$.

<u>Bade and Curtis (1960)</u> :

Let A be a strong Ditkin algebra, and let θ be an algebra homomorphism from A into a Banach algebra $\mathfrak{A}$. Then there is a <u>continuous</u> algebra hom. $\lambda : A \longrightarrow \mathfrak{A}$ such that $(\theta - \lambda)(A) \subseteq \mathfrak{S}(\theta)$.

Proof of 4.6.3 for strong Ditk. algebras.

Given a strong Ditkin algebra A and a Banach extension which splits, say

$$0 \longrightarrow I \xrightarrow{\iota} \mathfrak{A} \underset{\theta}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0$$

so $\theta : A \to \mathfrak{A}$ is an algebra hom. $A \to \mathfrak{A}$ with $\pi\theta = Id_A$.

We note $\pi(\mathfrak{S}(\theta)) = \{0\}$.

For if $a_n \in A$ with $a_n \to 0$ and $\theta a_n \to b$ in $\mathfrak{A}$, then $(\pi\theta) = Id_A$, so $(\pi\theta)a_n = a_n \to 0$.

But $\theta a_n \to b$, so $\pi\theta a_n \to \pi(b)$.

Thus $\pi(b) = 0$.

By Bade and Curtis, $\exists$

continuous algebra hom

$\lambda : A \to \mathfrak{A}$ with

$(\theta - \lambda)(A) \subseteq \mathfrak{R}(\theta)$.

So $\pi(\theta - \lambda) = 0$, and

$\pi\lambda = \pi\theta = Id_A$.

Thus λ is a continuous splitting hom.

$\square$