Normed algebras of differentiable functions on compact plane sets

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Abstract

These slides are available from the web page

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We solve some problems raised in an earlier paper of Bland and Feinstein (2005) by constructing a variety of compact plane sets X with dense interior such that $D^{(1)}(X)$ is not complete.

We also show that the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

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Definition

Let X be a perfect, compact plane set X and let $f \in C(X)$.

We say that f is **differentiable** at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\, z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We then call $f'(a)$ the (**complex**) **derivative** of f at a.

Using this concept of derivative, we define the terms **differentiable on** X and **continuously differentiable on** X in the obvious way, and we denote the set of continuously differentiable functions on X by $D^{(1)}(X)$.

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Bland and Feinstein gave an example of a rectifiable Jordan arc such that $D^{(1)}(X)$ is incomplete, and showed that $D^{(1)}(X)$ is incomplete whenever X has infinitely many components.

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Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \|\cdot\|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.

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Theorem

Let X be a perfect, compact plane set, and let A be the Banach function algebra $(D^{(1)}(X), \| \cdot \|)$.
Then the only characters on $D^{(1)}(X)$ are the evaluations at points of X.
In particular, every character on $(D^{(1)}(X), \| \cdot \|)$ is continuous.

Proof.

Let ϕ be a character on A, and set $w = \phi(Z)$, where Z is the coordinate functional (restricted to X in this setting).

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Take $f \in A$ with $f(w) = 0$.

Since

$$\lim_{z \to w,\, z \in X} \frac{f(z)}{z - w} = f'(w)\,,$$

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It is now easy to see that $f^3 = (Z - w1)g$ for a (unique) function $g \in D^{(1)}(X)$ (with $g(w) = g'(w) = 0$).

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The result follows. $\square$

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A **path** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a < b$; γ is a path **from** $\gamma(a)$ **to** $\gamma(b)$ with **endpoints** $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

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A path in $\mathbb{C}$ is **admissible** if it is rectifiable and has no constant subpaths.

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We now recall the standard definitions of regularity and uniform regularity for compact plane sets.

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Their proof is equally valid for pointwise regular, compact plane sets, so in fact $D^{(1)}(X)$ is complete whenever X is a finite union of pointwise regular, compact plane sets.

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Now suppose that U is dense in X (and hence, in particular, X is perfect).

Then $A^{(1)}(X)$ is the set of functions f in $A(X)$ such that $(f|_U)'$ extends continuously to the whole of X.

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This is too restrictive for our purposes, and instead we shall mostly work with the larger class of compact plane sets X for which the union of the images of all admissible rectifiable paths in X is dense in X.

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We shall often take $\mathcal{F}$ to be the set of all admissible paths in X.

In this case, $\mathcal{F}$ is effective if and only if the union of the images of all admissible paths in X is dense in X.

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For $f \in C(X)$, we say that $g \in C(X)$ is an $\mathcal{F}$**-derivative** of f if, for all $\gamma \in \mathcal{F}$, we have

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We define

$$\mathcal{D}^{(1)}_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

If $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}_{\mathcal{F}}^{(1)}(X)$ by f'. Moreover, this agrees with the usual derivative for functions in $D^{(1)}(X)$.

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Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

For $f \in \mathcal{D}_{\mathcal{F}}^{(1)}(X)$, set $\|f\| = |f|_X + |f'|_X$.

If $\mathcal{F}$ is an effective family of paths in a compact plane set X, then $\mathcal{F}$-derivatives are unique, and so we may denote the $\mathcal{F}$-derivative of a function $f \in \mathcal{D}_{\mathcal{F}}^{(1)}(X)$ by f'. Moreover, this agrees with the usual derivative for functions in $D^{(1)}(X)$.

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Theorem

Let X be a compact plane set, and let $\mathcal{F}$ be an effective family of paths in X.

Then $(\mathcal{D}_{\mathcal{F}}^{(1)}(X), \|.\|)$ is a Banach function algebra containing $D^{(1)}(X)$ isometrically as a subalgebra.

Proof of the completeness of $(\mathcal{D}^{(1)}_{\mathcal{F}}(X), \|.\|)$

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Thus $(\mathcal{D}^{(1)}_{\mathcal{F}}(X), \|\cdot\|)$ is complete. $\square$

The completion of $D^{(1)}(X)$

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}_{\mathcal{F}}^{(1)}(X)$.

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For X and $\mathcal{F}$ as in this theorem and proof, we do not know whether or not $\widetilde{D}^{(1)}(X)$ is always equal to $\mathcal{D}_{\mathcal{F}}^{(1)}(X)$.
In general there are easy examples where $\mathcal{F}$ is effective but $\widetilde{D}^{(1)}(X) \neq \mathcal{D}_{\mathcal{F}}^{(1)}(X)$.

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In particular, the polynomials are not dense in $A^{(1)}(X)$.

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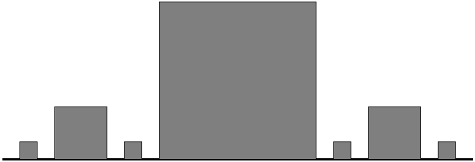

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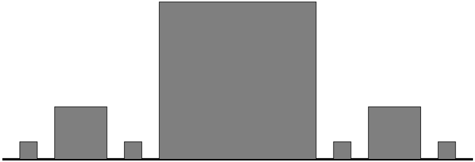

The Cantor function of x, regarded as a function of $z = x + \mathrm{i}y$, is then in $A^{(1)}(X) \setminus \mathcal{D}^{(1)}_{\mathcal{F}}(X)$.

We now return to the question of the completeness of $(D^{(1)}(X), \|\cdot\|)$.

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Theorem

Let X be a perfect, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $A_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z) - f(w)| \leq A_z(|f|_X + |f'|_X)|z - w|. \tag{1}$$

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Note that X need not be connected here. However, the condition implies that X has only finitely many components.

For pointwise regular X, (1) is certainly satisfied, and indeed the $|f|_X$ term may be omitted from the right-hand side of (1).

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We now show that this $|f|_X$ term may also be omitted under the weaker assumption that X be connected.

First we require a lemma concerning functions whose derivatives are constantly 0.

Lemma

Let X be a connected, compact plane set for which $(D^{(1)}(X), \|\cdot\|)$ is complete.

Let $f \in D^{(1)}(X)$ be such that $f' = 0$. Then f is a constant.

Proof. Assume towards a contradiction that there exists $f \in D^{(1)}(X)$ such that $f' = 0$ and such that f is not a constant.

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Consideration of the functions $g_n = 1 - f^n$ then quickly leads to the desired contradiction. □

We are now ready to eliminate the $|f|_X$ term from the right-hand side of equation (1) under the assumption that X is connected.

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Let X be a compact plane set, and let $z_0 \in X$. Then we define

$$M_{z_0}^{(1)}(X) = \{f \in D^{(1)}(X) : f(z_0) = 0\}$$

so that $M_{z_0}^{(1)}(X)$ is a maximal ideal in $D^{(1)}(X)$.

Theorem

Let X be a connected, compact plane set for which $(D^{(1)}(X), \|\cdot\|)$ is complete. Let $z_0 \in X$. Then there exists a constant $C_1 > 0$ such that, for all $f \in M_{z_0}^{(1)}(X)$, we have

$$|f|_X \leq C_1 \left|f'\right|_X .$$

Furthermore, there exists another constant $C_2 > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z_0) - f(w)| \leq C_2 \left|f'\right|_X |z_0 - w| . \qquad (2)$$

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Assume towards a contradiction that there is a sequence $(f_n) \in M_{z_0}^{(1)}(X)$ such that $|f_n|_X = 1$ for each $n \in \mathbb{N}$, but such that $|f_n'|_X \to 0$ as $n \to \infty$.

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Let $z \in X$. Since $(D^{(1)}(X), \|\cdot\|)$ is complete, there is a constant $C_z > 0$ such that

$$|f(z) - f(w)| \leq C_z(|f|_X + \left|f'\right|_X) |z - w| \leq 2C_z |w - z|$$

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Clearly we have $f(z_0) = 0$ and $|f|_X = 1$. We know that $|f_n'|_X \to 0$ as $n \to \infty$, and so (f_n) is a Cauchy sequence in $(D^{(1)}(X), \|\cdot\|)$.

Since $(D^{(1)}(X), \|\cdot\|)$ is complete, (f_n) is convergent in this space. Clearly $\lim_{n\to\infty} f_n = f$ in $D^{(1)}(X)$, and so $f' = 0$.

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By the preceding lemma, f is a constant. But $f(z_0) = 0$, and so $f = 0$, a contradiction of the fact that $|f|_X = 1$.

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We now set $C_2 = A_{z_0}(1 + C_1)$, where A_{z_0} is the constant from (1).

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Equation (2) now follows. $\square$

The above theorem does not hold in the absence of either of the hypotheses that X be connected or that $(D^{(1)}(X), \|\cdot\|)$ be complete.

The following corollary is now immediate.

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Corollary

Let X be a connected, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $B_z > 0$ such that, for all $f \in D^{(1)}(X)$ and all $w \in X$, we have

$$|f(z) - f(w)| \leq B_z \left|f'\right|_X |z - w| . \tag{3}$$

Let X be a polynomially convex, geodesically bounded compact plane set. Then X is connected, and we know that $P_0(X)$ is dense in $(D^{(1)}(X), \|\cdot\|)$.

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Corollary

Let X be a polynomially convex, geodesically bounded, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if, for each $z \in X$, there exists $B_z > 0$ such that, for all $p \in P_0(X)$ and all $w \in X$, we have

$$|p(z) - p(w)| \le B_z \left|p'\right|_X |z - w| \ . \tag{4}$$

We have no counterexample to the following conjecture.

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Let X be a connected, compact plane set. Then $(D^{(1)}(X), \|\cdot\|)$ is complete if and only if X is pointwise regular.

By a variety of methods, we have proved this conjecture for several classes of compact plane sets.

In particular, the conjecture holds for all rectifiably connected, polynomially convex compact plane sets with empty interior, for all star-shaped, compact plane sets, and for all Jordan arcs in $\mathbb{C}$ (rectifiable or not).

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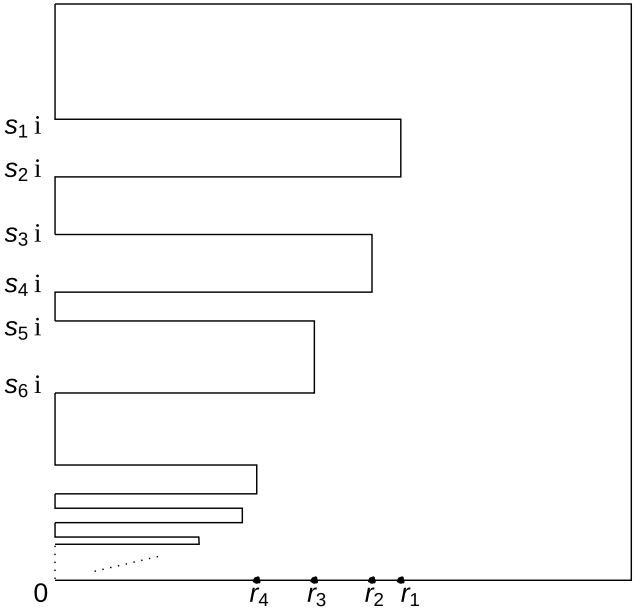
Indeed, Conjecture 1 holds for all sets of the type shown in the following diagram.


s_1 i
s_2 i
s_3 i
s_4 i
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s_6 i
0
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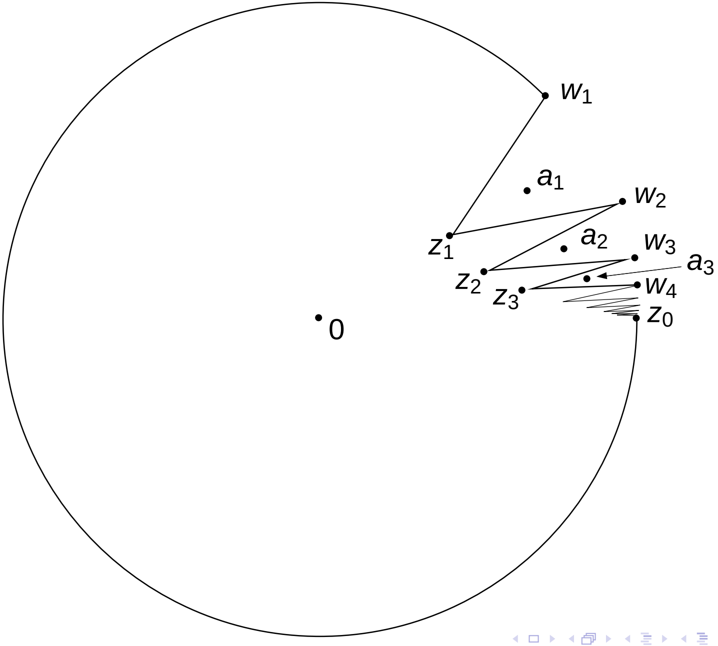
Radially self-absorbing, compact plane sets are always star-shaped, polynomially convex, geodesically bounded, and have dense interior.

Example

There exists a radially self-absorbing, compact plane set X such that $D^{(1)}(X)$ is incomplete.

This is not hard, given that Conjecture 1 holds for all star-shaped sets.



w_1
a_1
w_2
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We do not know whether or not Conjecture 1 holds for sets of the following type.

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