Introduction to compact operators

Joel Feinstein

October 31, 2007

1 Preliminary definitions and results concerning metric spaces

Definition 1.1 Let (X, d) be a metric space, let $x \in X$ and let $r > 0$. Then the **open ball in X centred on x and with radius** r, denoted by $B_X(x, r)$, is defined by

$$B_X(x, r) = \{y \in X : d(y, x) < r\}.$$

The corresponding **closed ball**, denoted by $\bar{B}_X(x, r)$, is defined by

$$\bar{B}_X(x, r) = \{y \in X : d(y, x) \leq r\}.$$

If there is no ambiguity over the metric space involved, we may write $B(x, r)$ and $\bar{B}(x, r)$ instead.

Definition 1.2 A metric space (X, d) is **complete** if **every** Cauchy sequence in X converges in X. Otherwise X is **incomplete**: this means that there is at least one Cauchy sequence in X which does not converge in X. We also describe the metric d as either a **complete metric** or an **incomplete metric**, accordingly.

Recall that every subset Y of a metric space X is also a metric space, using the restriction of the original metric on X.

It is standard that this restricted metric also induces the subspace topology on Y, so we may call this restriction the subspace metric.

Unless otherwise specified, we will always use the subspace metric/topology when discussing metric/topological properties of subsets of metric/topological spaces.

Definition 1.3 A metric space X is **sequentially compact** if every sequence in X has at least one convergent subsequence.

Definition 1.4 A metric space X is **totally bounded** if, for all $\varepsilon > 0$, X has a finite cover consisting of ε-balls.

Gap to fill in

Exercise. Let A be a subset of a metric space X. Show that (as metric spaces) A is totally bounded if and only if $\operatorname{clos} A$ (the closure of A) is totally bounded.

We will use without proof the following standard characterizations of compact metric spaces.

Proposition 1.5 Let X be a metric space. Then the following conditions on X are equivalent:

(a) X is compact;

(b) X is sequentially compact;

(c) X is complete and totally bounded.

We shall reformulate this result slightly, using the notion of a relatively compact subset of a topological space.

Definition 1.6 Let X be a topological space and let A be a subset of X. Then A is **relatively compact** if $\operatorname{clos} A$ is a compact subset of X.

Gap to fill in

It is now an exercise to check the following reformulation of the result above.

Proposition 1.7 Let X be a complete metric space and let A be a subset of X. Then the following conditions on A are equivalent.

(a) A is relatively compact;

(b) every sequence in A has a subsequence which converges in X;

(c) A is totally bounded.

2 Bounded operators and compact operators

Let X be a Banach space. We denote by $\mathcal{B}(X)$ the Banach space of all bounded (continuous) linear operators from X to X, with operator norm $\|\cdot\|_{\mathrm{op}}$.

Definition 2.1 With X as above, let $T \in \mathcal{B}(X)$. Then T is a **compact operator** (or T **is compact**) if $T(\bar{B}_X(0,1))$ is a relatively compact subset of X.

Examples and non-examples. Compact operators are everywhere! We mention a few easy examples and non-examples.

(a) Finite-rank operators are compact.

Gap to fill in

(b) In particular, every linear operator on a finite-dimensional Banach space is compact.

Gap to fill in

(c) Norm limits of finite-rank operators are compact.

Gap to fill in

(d) Suppose that X is an infinite-dimensional Banach space. Then the identity operator I on X is not compact and, more generally, no invertible bounded linear operator on X can be compact.

Gap to fill in

In view of our earlier characterization of relatively compact subsets of complete metric spaces, we have the following result for operators.

Proposition 2.2 Let X be a Banach space, and let $T \in \mathcal{B}(X)$. Then the following conditions on T are equivalent:

(a) T is compact;

(b) $T(\bar{B}(0,1))$ is totally bounded;

(c) for every bounded sequence $(x_n) \subseteq X$, the sequence $(T(x_n))$ has a convergent subsequence.

We shall denote the set of compact operators on X by $\mathcal{K}$ (or $\mathcal{K}(X)$).

Theorem 2.3 With X and $\mathcal{K}$ as above, $\mathcal{K}$ is a closed, two-sided ideal in the Banach algebra $\mathcal{B}(X)$.

Gap to fill in

2.1 Invertibility and spectra

From now on, X will always be a **complex** Banach space. Recall the Banach Isomorphism Theorem.

Theorem 2.4 Let $T \in \mathcal{B}(X)$. Suppose that T is bijective (so that T is a linear isomorphism from X to X). Then $T^{-1} \in \mathcal{B}(X)$, and T is a linear homeomorphism.

Thus there is no ambiguity in discussing the issue of invertibility for bounded linear operators on Banach spaces, and we see that this coincides with the notion of invertibility in the Banach algebra $\mathcal{B}(X)$.

Definition 2.5 Let $T \in \mathcal{B}(X)$. Then the **spectrum** of T, $\sigma(T)$, is defined to be the set

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}\,.$$

It is standard that $\sigma(T)$ is always a non-empty, compact subset of $\mathbb{C}$.

In the case of operators on finite-dimensional spaces, the spectrum is simply the set of eigenvalues of the operator.

Definition 2.6 Let $T \in \mathcal{B}(X)$. Then the **spectral radius** of T, $\rho(T)$, is defined by

$$\rho(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}\,.$$

The following result is the famous **spectral radius formula**.

Theorem 2.7 Let $T \in \mathcal{B}(X)$. Then

$$\rho(T) = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

Gap to fill in

Example 2.8 Consider the special case where $X = \ell_2$. Let (a_n) be any bounded sequence of complex numbers. Then we may define $T \in \mathcal{B}(X)$ by

$$T((x_n)) = (a_1 x_1, a_2 x_2, a_3 x_3, \dots) .$$

The eigenvalues of T are then precisely the complex numbers a_n, and $\sigma(T)$ is the closure in $\mathbb{C}$ of this set of eigenvalues.

In this setting, T is compact if and only if the sequence (a_n) converges to 0.

Gap to fill in

3 Spectra and eigenspaces of compact operators

Throughout this section, T is a compact operator on an infinite-dimensional complex Banach space, X.

In this setting, it is easy to check that if Y is a closed subspace of X such that $T(Y) \subseteq Y$ (i.e. Y is an **invariant** subspace for T) then $T|_Y$ is a compact operator on Y.

Gap to fill in

Since T can not be invertible, we know that $0 \in \sigma(T)$.

The above example suggests that there may be some other restrictions on the spectrum that T can have.

Theorem 3.1 Suppose that λ is a non-zero element of $\sigma(T)$. Then λ is an eigenvalue of T, and the eigenspace $\ker(T - \lambda I)$ is finite-dimensional.

Gap to fill in

Finally, we state the main result concerning spectra of compact operators.

Theorem 3.2 If $\sigma(T) \setminus \{0\}$ is an infinite set, then it may be written as a sequence (λ_n) which converges to 0. We may arrange for this sequence to be non-increasing in modulus, and to include each non-zero eigenvalue with the appropriate multiplicity.

Gap to fill in