Introduction to Fredholm operators II

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1 Revision

As last time, X will be an **infinite-dimensional, complex** Banach space.

We denote by $\mathcal{B}$ (or $\mathcal{B}(X)$) the Banach algebra of all bounded (continuous) linear operators from X to X, with operator norm $\| \cdot \|_{\mathrm{op}}$.

We denote the set of compact linear operators on X by $\mathcal{K}$ (or $\mathcal{K}(X)$).

We denote by $\mathcal{F}$ (or $\mathcal{F}(X)$) the set of all finite-rank linear operators on X.

Theorem 1.1 With X and $\mathcal{K}$ as above, $\mathcal{K}$ is a closed, two-sided ideal in the Banach algebra $\mathcal{B}$.

We form the quotient Banach algebra $\mathcal{B}(X)/\mathcal{K}(X) = \mathcal{B}/\mathcal{K}$, with the quotient norm given by

$$\|T + \mathcal{K}(X)\| = \mathrm{dist}(T, \mathcal{K}(X)) = \inf_{K \in \mathcal{K}(X)} \|T - K\|_{\mathrm{op}} .$$

This unital Banach algebra is called the **Calkin algebra**.

Let $T \in \mathcal{B}$. Then T is a **Fredholm operator** if both the kernel of T is finite-dimensional and the image of T, $\operatorname{im} T$, has finite codimension in X: $\dim(\ker T) < \infty$ and $\dim(X/\operatorname{im} T) < \infty$.

Proposition 1.2 Let $T \in \mathcal{B}$ be a Fredholm operator. Then $T(X)$ is a closed subspace of X.

It then follows easily that, for a Fredholm operator T, both $\ker T$ and $T(X)$ are **complemented** in X (in the sense of Banach spaces).

2 Fredholmness conditions revisited

Last time we stated, without proof, the following result.

Theorem 2.1 Let $T \in \mathcal{B}$. Then the following statements are equivalent:

(a) T is a Fredholm operator;

(b) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are finite-rank;

(c) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are compact;

(d) $T + \mathcal{K}$ is invertible in the Calkin algebra;

(e) $T + \mathcal{F}$ is invertible in the algebra $\mathcal{B}/\mathcal{F}$;

(f) $T + \bar{\mathcal{F}}$ is invertible in $\mathcal{B}/\bar{\mathcal{F}}$.

Here (b) is clearly equivalent to (e), and (c) is equivalent to (d). It is also easy to see that (e)⇒(f)⇒(d).

The implication (a) ⇒(b) is a relatively easy exercise involving the projections onto the complemented subspaces discussed above (see Heuser).

This then leaves the non-trivial fact that (c) $\Rightarrow$(a). This follows easily once you can prove that $I - K$ is a Fredholm operator whenever $K \in \mathcal{K}$.

In the next two sections we look at some of the results on operators which can be used to prove this implication. (There may be a quicker way?)

3 Chains of subspaces associated with linear operators

Let E be any vector space, and let T be a linear operator from E to E (a **linear endomorphism** of E).

As n increases, the kernels of the linear operators T^n are nested increasing, while the images $\mathrm{im}(T^n) = T^n(E)$ are nested decreasing.

These 'chains' of subspaces may or may not 'stabilize'.

Moreover, one may stabilize without the other doing so.

Gap to fill in

This is the "null chain" of T.

$\{0\} \subseteq \ker T \subseteq \ker T^2 \subseteq \ker T^3 \subseteq \dots$

$E \supseteq TE \supseteq T^2(E) \supseteq \dots$

image chain

If $T^n(E) = T^{n+1}(E)$

then $T^{n+k}(E) = T(E)$ for all $k \in \mathbb{N}$.

Kernel is similar.

Example: right shift and left shift on ℓ_2.

$R((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, \dots)$

$S((x_1, x_2, x_3, \dots)) = (x_2, x_3, x_4, \dots)$

R^n injective all n, so

null chain stabilizes immediately for R.

But image chain $R^n(\ell_2)$ never stabilizes.

S is surjective: image chain stabilizes immediately, but null chain never stabilizes.

A remarkable fact is that, **if both chains stabilize**, they must first do so at the same value at n, and for that n (or larger) we have

$$E = \ker(T^n) \oplus \operatorname{im}(T^n)\,.$$

In this case we say that the operator T is **chain-finite** with **chain length** equal to this minimal value of n

The following result may be used to prove the above fact.

Lemma 3.1 Let S be a linear endomorphism of the vector space E. Then the following hold:

(a) $\ker(S^2) = \ker S \Leftrightarrow \ker S \cap S(E) = \{0\}\,;$

(b) $S^2(E) = S(E) \Leftrightarrow \ker S + S(E) = E\,;$

(c) $E = \ker S \oplus S(E) \Leftrightarrow$ both $\ker(S^2) = \ker S$ and $S^2(E) = S(E)\,.$

Gap to fill in

(c) Is immediate from (a) and (b).

(a) ($\Rightarrow$) Given $\ker(S^2) = \ker S$.

Let $x \in \ker S \cap S(E)$. Then $x = S(y)$ for some $y \in E$. Then $S^2(y) = Sx = 0$ since $x \in \ker S$. So $y \in \ker(S^2) = \ker S$.
Thus $x = Sy = 0$.

($\Leftarrow$) If $\ker S \cap T(E) = \{0\}$, we show $\ker(S^2) = \ker(S)$.
Let $y \in \ker(S^2)$. We show $y \in \ker S$.
Set $x = Sy$. Then $Sx = 0$, so $x \in \ker(S) \cap S(E) = \{0\}$.
So $Sy = x = 0$, and $y \in \ker S$.
Rest is easy.

(b) ($\Rightarrow$) Suppose $S(E) = S^2(E)$.
Claim: $\ker S + S(E) = E$.
Let $x \in E$. Then $Sx \in S(E) = S^2(E)$, so $\exists\, y \in E$ with $Sx = S^2 y$.
This gives $S(x - Sy) = 0$.
Thus $x - Sy \in \ker S$, and
$x = (x - Sy) + Sy \in \ker S + S(E)$.

($\Leftarrow$) Given $E = \ker S + S(E)$, show $S(E) = S^2(E)$.

Let $x \in E$. We show $Sx \in S^2(E)$.

We can write $x = w + S(y)$ for some $w \in \ker S$, $y \in E$.

Then $Sx = S^2 y$. $\square$

For interest, see also "Fitting's Lemma" (Google/Wikipedia)

4 The Riesz–Schauder theory of compact operators

Let K is a compact operator on X, and set $T = I - K$.

We wish to show that T is a Fredholm operator.

As a good start, note that $\ker T$ is the eigenspace of K corresponding to eigenvalue 1, and so must be finite-dimensional. Using the binomial expansion, we see that T^n is an operator of the same type, and so $\ker(T^n)$ is also finite-dimensional.

Gap to fill in

$$(I-K)^n = I^n - nI^{n-1}K + \cdots$$
$$= I - K(\cdots)$$

Since $\mathcal{K}$ is an ideal, RHS is I - compact.

$T = I - K.$

Lemma 4.1 With T as above, $T^n(X)$ is closed for all $n \in \mathbb{N}$.

Gap to fill in

Only need case $n=1$, since

T^n also has form I − compact.

We prove that TX is closed.

Note: $T + K = I$

$Tx + Kx = x$ all x in X.

Let $(y_n) \subseteq TX$ and

suppose that $y_n \to y$ in X.

We show that $y \in TX$.

Choose x_n in X with $Tx_n = y_n$.

Further, by subtracting off

suitable elements of $\ker T$,

WMA that $\|x_n\| \leq 2\,\mathrm{dist}(x_n, \ker T)$.

Claim: $\|x_n\|$ is bounded under these circumstances.

Suppose not, for contradiction. Then there is a subsequence x_{n_k} with $\|x_{n_k}\| \to \infty$ as $k \to \infty$.

and such that $x_{n_k} \neq 0$ all k

Set $z_k = \dfrac{x_{n_k}}{\|x_{n_k}\|}$. $\|z_k\| = 1$.

Since $Tx_{n_k} \to y$, $\boxed{Tz_k \to 0}$

(since $\|x_{n_k}\| \to \infty$).

$z_k = Tz_k + Kz_k$. (*)

Kz_k has a convergent subsequence.

WMA $Kz_k \to w$ as $k \to \infty$.

Then $z_k \to w$ as $k \to \infty$.

(by (*)).

Then $Tz_k \to 0$, so $Tw = 0$

and $w \in \ker T$.

But $\|x_{n_k}\| \leq 2 \operatorname{dist}(x_{n_k}, \ker T)$

so $\operatorname{dist}(z_k, \ker T) \geq \frac{1}{2}$ all k.

$\Rightarrow\!\Leftarrow$ So (x_n) are bounded.

We have

$$x_n = Tx_n + Kx_n .$$

We have $Tx_n \to y$.

(x_n) bounded, so Kx_n has a convergent subsequence, say

$$Kx_{n_k} \to v \in X.$$

Then $x_{n_k} = Tx_{n_k} + Kx_{n_k}$

$$\to y + v \quad \text{as } k \to \infty.$$

Set $x = y + v$.

Then $Tx_{n_k} \to Tx$.

This gives $Tx = y$. $\square$

The proof of the next theorem is based on the Riesz Geometric Lemma. (Apparently, this is the application for which the lemma was invented!)

Gap to fill in

Riesz Geometric Lemma:

If E is a normed space and F is a proper closed subspace of E, then, for all $\varepsilon > 0$, there is an $x \in E$ with $\|x\| = 1$ and

$\operatorname{dist}(x, F) > 1 - \varepsilon$.

$T = I - K$, K compact

Theorem 4.2 With T as above, both of the chains of subspaces $\ker(T^n)$ and $T^n(X)$ stabilize.

null chain image chain

Once we have proved this, we know that there is an $n \in \mathbb{N}$ such that $X = \ker(T^n) \oplus T^n(X)$.

Then $T^n(X)$ has finite-codimension in X, and the same holds for T. Thus T is a chain-finite Fredholm operator.

Gap to fill in

See Heuser's book on Functional Analysis.

5 The index of a Fredholm operator

For proofs of the results in this section, we refer the reader to Heuser's book.

Definition 5.1 Let T be a Fredholm operator on X. Then the **index** $\operatorname{ind} T$ of T is defined by

$$\operatorname{ind} T = \dim(\ker T) - \dim(X/\operatorname{im} T)\,.$$

Examples.

Gap to fill in

R, S as before on ℓ_2.

$\dim(\ker S) = 1$, $\operatorname{codim}(\operatorname{im} S) = 0$

so $\operatorname{ind}(S) = 1$.

$\operatorname{ind}(R) = -1$.

$\operatorname{ind}(T) = 0$ if T is invertible.

$\operatorname{ind}(RS) = \operatorname{ind}(SR) = 0$.

The following result shows that index is **additive**, in an appropriate sense.

Proposition 5.2 Let S and T be Fredholm operators on X. Then so is ST, and

$$\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T\,.$$

We discussed chain-finite Fredholm operators above. Here we have the following result.

Theorem 5.3 Let T be a chain-finite Fredholm operator on X. Then $\operatorname{ind} T = 0$. Moreover, T can be written in the form $R + \cancel{K}\,S$, where $R \in \mathcal{B}$ is invertible, $S \in \mathcal{F}$ and $RS = SR$. Conversely, any operator of the latter form is a chain-finite Fredholm operator.

Index is unaffected by the addition of a compact operator.

Proposition 5.4 Let T be a Fredholm operator. Then, for every $K \in \mathcal{K}$, $T + K$ is also a Fredholm operator, and

$$\operatorname{ind}(T + K) = \operatorname{ind}(T)\,.$$

see below for a special case.

Finally, we return to discussion of the Fredholm region. Recall the following definition from last time.

N.b. $\operatorname{ind}(I) = 0$.

$I + K$ is chain-finite. So $\operatorname{ind}(I+K) = 0$.

Definition 5.5 Let $T \in \mathcal{B}$. Then the **essential spectrum** of T, $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Fredholm operator}\}.$$

Compact, and $\subseteq \sigma(T)$.

The complement of this set,

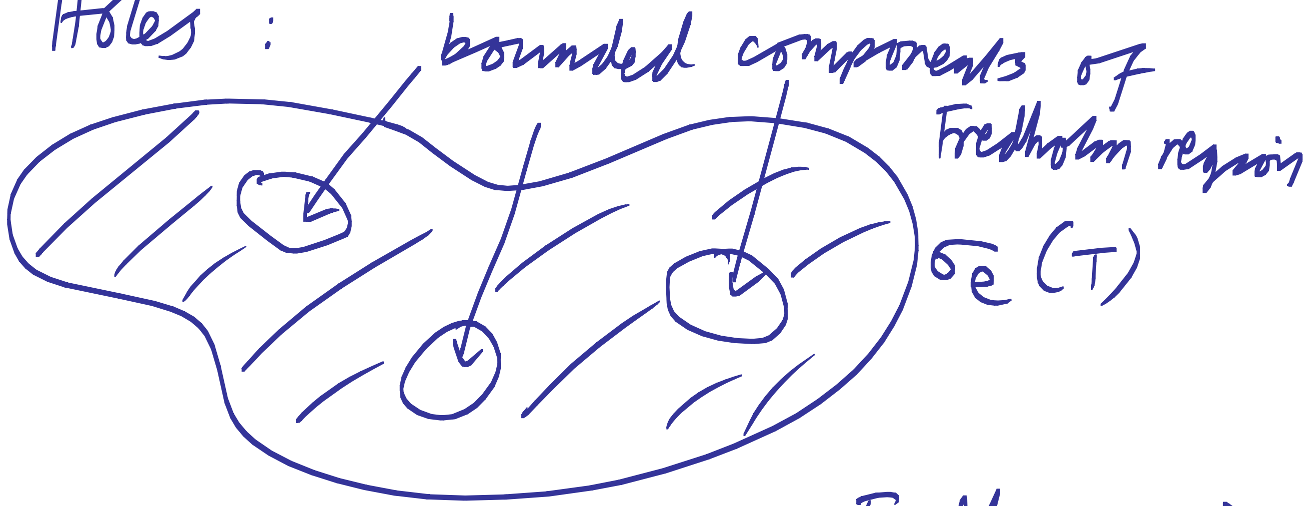
$$\mathbb{C} \setminus \sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is a Fredholm operator}\},$$

is the **Fredholm region** for T.

The Fredholm region is open, and has exactly one unbounded component.

Gap to fill in

"Holes":



Unbounded component of Fredholm region.

Theorem 5.6 Let V be a component of the Fredholm region for T. Then $\operatorname{ind}(\lambda I - T)$ is the same for all $\lambda \in V$. Moreover, exactly one of the following holds for V.

(a) For all $\lambda \in V$, the null chain for $\lambda I - T$ stabilizes. In this case, the set of eigenvalues of T in V (if any) has no cluster point in V.

OR

(b) For all $\lambda \in V$, the null chain for $\lambda I - T$ does NOT stabilize. In this case, every point of V is an eigenvalue for T.

Gap to fill in

Case (a): may be no eigenvalues in V, or finitely many, or infinitely many: in the last case, the only accumulation points of these eigenvalues are in $\sigma_e(T)$.

Consider $S: \ell_2 \to \ell_2$.

Then $0 \in \sigma(S)$. $0 \notin \sigma_e(S)$

[S is Fredholm but not invertible.]

unit circle

$\sigma_e(S) = \mathbb{T}$. (Check this!)

Since null chain for

$-S = \underset{\lambda=0}{0}I - S$ does not

stabilize, we are in case (b).

Thus $\lambda I - S$ has non-stabilizing

null chain for all λ in that

component of the Fredholm region,

i.e. $\{z \in \mathbb{C} \mid |z| < 1\}$.