See end of annotations for some ADDITIONAL COMMENTS added later.

Introduction to Fredholm operators

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1 Preliminary definitions, terminology, notation and results

Throughout this talk, X will be an **infinite-dimensional, complex** Banach space.

We denote by $\mathcal{B}$ (or $\mathcal{B}(X)$) the Banach algebra of all bounded (continuous) linear operators from X to X, with operator norm $\|\cdot\|_{\mathrm{op}}$.

Gap to fill in

Banach algebra $(A, \|\cdot\|)$ is a complex algebra with a complete norm $\|\cdot\|$ which is submultiplicative: $\|ab\| \le \|a\|\|b\|$.

Example. Let Y be a compact Hausdorff space.

$$C(Y) = \{ f : Y \to \mathbb{C} \mid f \text{ is cts} \},$$

$$\|f\|_\infty = \sup_{y \in Y} |f(y)| .$$

$(C(Y), \|\cdot\|_\infty)$ is a Banach algebra.

Unital Banach algebras have an identity element 1 with $\|1\| = 1$.

We denote the set of compact linear operators on X by $\mathcal{K}$ (or $\mathcal{K}(X)$). $\mathcal{K}(X)$ or $\mathcal{K}$

We denote by $\mathcal{F}$ (or $\mathcal{F}(X)$) the set of all finite-rank linear operators on X.

Recall the following result from last time.

Theorem 1.1 With X and $\mathcal{K}$ as above, $\mathcal{K}$ is a closed, two-sided ideal in the Banach algebra $\mathcal{B}$.

Because X is infinite-dimensional here, we have $\mathcal{K}(X) \neq \mathcal{B}(X)$.

We may form the quotient Banach algebra $\mathcal{B}(X)/\mathcal{K}(X) = \mathcal{B}/\mathcal{K}$, with the quotient norm given by

$$\|T + \mathcal{K}(X)\| = \operatorname{dist}(T, \mathcal{K}(X)) = \inf_{K \in \mathcal{K}(X)} \|T - K\|_{\text{op}} .$$

This unital Banach algebra is called the **Calkin algebra**, and is very important in the theory.

We know that $\mathcal{F} \subseteq K^{\mathcal{K}}$, and so $\operatorname{clos} \mathcal{F} = \bar{\mathcal{F}} \subseteq K$. $\mathcal{K}$

It is easy to check that $\mathcal{F}$ is a 2-sided ideal in $\mathcal{B}$, but that $\mathcal{F}$ is not closed.

For 'nice' Banach spaces, $\bar{\mathcal{F}} = \mathcal{K}$, but this is not always the case.

However, $\bar{\mathcal{F}}$ is always a closed 2-sided ideal in $\mathcal{B}$.

2 Invertibility and spectra

Recall the following definition from last time.

Definition 2.1 Let $T \in \mathcal{B}(X)$. Then the **spectrum** of T, $\sigma(T)$, is defined to be the set

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

or $\lambda I - T$

We generalize this definition to unital Banach algebras A.

Definition 2.2 Let A be a unital Banach algebra with identity element 1, and let $a \in A$. Then the **spectrum** of a, $\sigma(a)$, is defined to be the set

$$\{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}.$$

It is standard that $\sigma(a)$ is always a non-empty, compact subset of $\mathbb{C}$.

Gap to fill in

The spectrum of an operator T in $\mathcal{B}(X)$ is as before.

In $C(Y)$ (as above), $\sigma(f) = f(Y)$.

As for operators, the **spectral radius** of a, which we now denote by $r(a)$, is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}\,.$$

The **spectral radius formula** then tells us that.

$$r(a) = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

We will mainly be interested in the Banach algebras $\mathcal{B}$ and the Calkin algebra.

3 Definition and examples of Fredholm operators

Let $T \in \mathcal{B}$. Then T is a **Fredholm operator** if both the kernel of T is finite-dimensional and the image of T has finite codimension in X: $\dim(\ker T) < \infty$ and $\dim(X/T(X)) < \infty$. T has "finite deficiency".

Examples. Obviously, every invertible operator in $\mathcal{B}$ is a Fredholm operator.

Gap to fill in

Consider $X = \ell_2$.

$= \{ x = (x_n) \subseteq \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty \}$

with norm $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$.

$R =$ right shift

$S =$ left shift

With $x = (x_n)$ as above

$R(x) = (0, x_1, x_2, x_3, \ldots)$

$S(x) = (x_2, x_3, x_4, \ldots)$.

$SR(x) = x$ all x

$RS(x) = (0, x_2, x_3, x_4, \ldots)$.

$\ker(R) = \{0\}$

$R(\ell_2) = \{ x \in \ell_2 \mid x_1 = 0 \}$.

This has codimension 1. So

R is Fredholm.

$\ker(S)$ is 1-dimensional.

S is surjective, so $S(\ell_2)$ has codim 0. S is Fredholm too.

The following facts are not obvious. We give some proofs: for the rest we refer the reader to, for example, Heuser's book on Functional Analysis.

Proposition 3.1 Let $T \in \mathcal{B}$ be a Fredholm operator. Then $T(X)$ is a closed subspace of X.

It then follows easily that, for a Fredholm operator T, both $\ker T$ and $T(X)$ are **complemented** in X (in the sense of Banach spaces).

Gap to fill in

In fact we have:

Lemma

If $T \in \mathcal{B}$ and X has a closed subspace E such that

$$T(X) \oplus E = X.$$

Then $T(X)$ must be closed.

The result above then follows from this.

Proof of Lemma.

We have $T(X) \oplus E = X$.

Look at Banach space external direct sum $X \oplus E$; norm $\|(x, y)\| = \|x\| + \|y\|$ $(x \in X, y \in E)$.

Define $S : X \oplus E \to X$ by $S((x, y)) = Tx + y$.

Clearly S is bounded linear $X \oplus E \to X$.

By above, S is surjective, and so S is an open mapping. From this, easy to show TX is closed. $\square$

Erratum in podcast!

I said that the image of a closed subspace under (a linear operator S which is) an open mapping must be closed.

This is ~~not~~ always true!

[Tricky exercise: find a counterexample!]

It is true, however, if the closed subspace contains the kernel of S, as it does here.

[<u>Easier</u>: by quotienting out by $\ker T$ at the start, you can assume that T is injective. It then follows that S is a Banach space isomorphism, from which the rest is even easier.]

This leads on to another somewhat unintuitive result.

Theorem 3.2 Let $T \in \mathcal{B}$. Then the following statements are equivalent:

(a) T is a Fredholm operator;

or $S_1, S_2 \in \mathcal{B}$ $S_1 T - I$ and $T S_2 - I$

(b) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are finite-rank;

(or S_1, S_2 as above)

(c) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are compact;

(d) $T + \mathcal{K}$ is invertible in the Calkin algebra;

(e) $T + \mathcal{F}$ is invertible in the algebra $\mathcal{B}/\mathcal{F}$;   This quotient is not a normed alg.

(f) $T + \bar{\mathcal{F}}$ is invertible in $\mathcal{B}/\bar{\mathcal{F}}$.

Note that if T is a Fredholm operator and $K \in \mathcal{K}$ then $T + K$ is also a Fredholm operator.

In particular, the sum of an invertible operator and a compact operator is always a Fredholm operator.

4 The essential spectrum and the Fredholm region

The reader should be warned that the standard definition of the **essential spectrum** given here is **not** the same as the (non-standard) definition given in Heuser's book on Functional Analysis. Fortunately, this does not affect the value of the essential spectral radius, defined below.

Definition 4.1 Let $T \in \mathcal{B}$. Then the **essential spectrum** of T, $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Fredholm operator}\}.$$

In view of our earlier results, this is also equal to both the spectrum of $T + \mathcal{K}$ in the Calkin algebra $\mathcal{B}/\mathcal{K}$ and the spectrum of $T + \bar{\mathcal{F}}$ in $\mathcal{B}/\bar{\mathcal{F}}$.

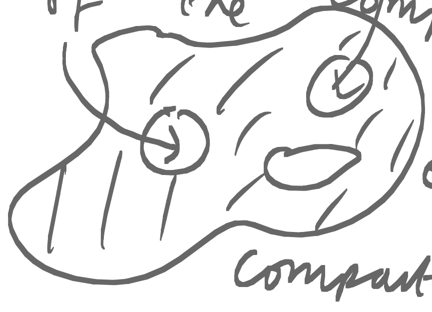
The complement of this set,

$$\mathbb{C} \setminus \sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is a Fredholm operator}\},$$

is the **Fredholm region** for T.

'holes' or bounded components of the complement.

Gap to fill in



$\sigma_e(T)$

compact $\subseteq \mathbb{C}$

Complement has exactly one unbounded component

The **essential spectral radius** of T, $r_e(T)$, is defined by

$$r_e(T) = \sup_{\lambda \in \sigma_e(T)} |\lambda| \, .$$

This is also equal to the spectral radius of $T + \mathcal{K}$ in the Calkin algebra and also of $T + \bar{\mathcal{F}}$ in $\mathcal{B}/\bar{\mathcal{F}}$.

Thus

$$r_e(T) = \inf_{n \in \mathbb{N}} \operatorname{dist}(T^n, \mathcal{K})^{1/n} = \lim_{n \to \infty} \operatorname{dist}(T^n, \mathcal{K})^{1/n}$$

$$= \inf_{n \in \mathbb{N}} \operatorname{dist}(T^n, \mathcal{F})^{1/n} = \lim_{n \to \infty} \operatorname{dist}(T^n, \mathcal{F})^{1/n} \, .$$

Gap to fill in

Note: if $\exists n \in \mathbb{N}$ with $\operatorname{dist}(T^n, \mathcal{K}) < 1$, then $r_e(T) < 1$. (Converse also true).

5 Some important classes of operators

Here we introduce several classes of operators.

Definition 5.1 Let $T \in \mathcal{B}$. We say that T is a **Riesz operator** if $\lambda I - T$ is Fredholm for all non-zero complex numbers λ. (Compact $\Rightarrow$ Riesz: invertible + compact is Fred.)

Thus T is Riesz if and only if $r_e(T) = 0$. We say that T is **quasicompact** if $r_e(T) < 1$. (*)

Thus T is quasicompact if and only if there exists $n \in \mathbb{N}$ with $\mathrm{dist}(T^n, \mathcal{K}) < 1$.

We say that T is power compact if there exists a positive integer N such that T^N is compact.

Certainly we have the following implications:

compact $\Longrightarrow$ power compact $\Longrightarrow$ Riesz $\Longrightarrow$ quasicompact. In general, none of these implications can be reversed.

Gap to fill in

(*) This definition of quasicompact is the one given in Heuser's book (Page 216, exercise 4, English translation)

Apparently many authors use the term quasi-compact to mean power compact instead.

[The relevant paper of Feinstein and Kamowitz uses the weaker condition as in Heuser.]

<u>Note</u>. At the end of the audio podcast, I started to say "quasinilpotent" when I really meant "nilpotent".

[Nilpotent $\Longrightarrow$ quasinilpotent.

$\not\Leftarrow$

"quasinilpotent" means that the spectral radius is 0.]

A useful property of a bounded linear operator T on a Banach space is that each spectral element of T which lies in the unbounded component of the complement of the essential spectrum of T is an eigenvalue of finite multiplicity. Further, if there are infinitely many of them, then they cluster only on the essential spectrum.

We will discuss the relevant theory in more detail in a future session.

Gap to fill in