Introduction to Fredholm operators

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November 14, 2007

1 Preliminary definitions, terminology, notation and results

Throughout this talk, X will be an **infinite-dimensional, complex** Banach space.

We denote by $\mathcal{B}$ (or $\mathcal{B}(X)$) the Banach algebra of all bounded (continuous) linear operators from X to X, with operator norm $\|\cdot\|_{\mathrm{op}}$.

Gap to fill in

We denote the set of compact linear operators on X by $\mathcal{K}$ (or $\mathcal{K}(X)$).

We denote by $\mathcal{F}$ (or $\mathcal{F}(X)$) the set of all finite-rank linear operators on X.

Recall the following result from last time.

Theorem 1.1 With X and $\mathcal{K}$ as above, $\mathcal{K}$ is a closed, two-sided ideal in the Banach algebra $\mathcal{B}$.

Because X is infinite-dimensional here, we have $\mathcal{K}(X) \neq \mathcal{B}(X)$.

We may form the quotient Banach algebra $\mathcal{B}(X)/\mathcal{K}(X) = \mathcal{B}/\mathcal{K}$, with the quotient norm given by

$$\|T + \mathcal{K}(X)\| = \operatorname{dist}(T, \mathcal{K}(X)) = \inf_{K \in \mathcal{K}(X)} \|T - K\|_{\mathrm{op}} \,.$$

This unital Banach algebra is called the **Calkin algebra**, and is very important in the theory.

We know that $\mathcal{F} \subseteq \mathcal{K}$, and so $\operatorname{clos} \mathcal{F} = \bar{\mathcal{F}} \subseteq \mathcal{K}$.

It is easy to check that $\mathcal{F}$ is a 2-sided ideal in $\mathcal{B}$, but that $\mathcal{F}$ is not closed.

For 'nice' Banach spaces, $\bar{\mathcal{F}} = \mathcal{K}$, but this is not always the case.

However, $\bar{\mathcal{F}}$ is always a closed 2-sided ideal in $\mathcal{B}$.

2 Invertibility and spectra

Recall the following definition from last time.

Definition 2.1 Let $T \in \mathcal{B}(X)$. Then the **spectrum** of T, $\sigma(T)$, is defined to be the set

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

We generalize this definition to unital Banach algebras A.

Definition 2.2 Let A be a unital Banach algebra with identity element 1, and let $a \in A$. Then the **spectrum** of a, $\sigma(a)$, is defined to be the set

$$\{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}.$$

It is standard that $\sigma(a)$ is always a non-empty, compact subset of $\mathbb{C}$.

Gap to fill in

As for operators, the **spectral radius** of a, which we now denote by $r(a)$, is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}\,.$$

The **spectral radius formula** then tells us that.

$$r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

We will mainly be interested in the Banach algebras $\mathcal{B}$ and the Calkin algebra.

3 Definition and examples of Fredholm operators

Let $T \in \mathcal{B}$. Then T is a **Fredholm operator** if both the kernel of T is finite-dimensional and the image of T has finite codimension in X: $\dim(\ker T) < \infty$ and $\dim(X/T(X)) < \infty$.

Examples. Obviously, every invertible operator in $\mathcal{B}$ is a Fredholm operator.

Gap to fill in

The following facts are not obvious. We give some proofs: for the rest we refer the reader to, for example, Heuser's book on Functional Analysis.

Proposition 3.1 Let $T \in \mathcal{B}$ be a Fredholm operator. Then $T(X)$ is a closed subspace of X.

It then follows easily that, for a Fredholm operator T, both $\ker T$ and $T(X)$ are **complemented** in X (in the sense of Banach spaces).

Gap to fill in

This leads on to another somewhat unintuitive result.

Theorem 3.2 Let $T \in \mathcal{B}$. Then the following statements are equivalent:

(a) T is a Fredholm operator;

(b) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are finite-rank;

(c) There is an $S \in \mathcal{B}$ such that both $ST - I$ and $TS - I$ are compact;

(d) $T + \mathcal{K}$ is invertible in the Calkin algebra;

(e) $T + \mathcal{F}$ is invertible in the algebra $\mathcal{B}/\mathcal{F}$;

(f) $T + \bar{\mathcal{F}}$ is invertible in $\mathcal{B}/\bar{\mathcal{F}}$.

Note that if T is a Fredholm operator and $K \in \mathcal{K}$ then $T + K$ is also a Fredholm operator.

In particular, the sum of an invertible operator and a compact operator is always a Fredholm operator.

4 The essential spectrum and the Fredholm region

The reader should be warned that the standard definition of the **essential spectrum** given here is **not** the same as the (non-standard) definition given in Heuser's book on Functional Analysis. Fortunately, this does not affect the value of the essential spectral radius, defined below.

Definition 4.1 Let $T \in \mathcal{B}$. Then the **essential spectrum** of T, $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Fredholm operator}\}.$$

In view of our earlier results, this is also equal to both the spectrum of $T + \mathcal{K}$ in the Calkin algebra $\mathcal{B}/\mathcal{K}$ and the spectrum of $T + \bar{\mathcal{F}}$ in $\mathcal{B}/\bar{\mathcal{F}}$.

The complement of this set,

$$\mathbb{C} \setminus \sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is a Fredholm operator}\},$$

is the **Fredholm region** for T.

Gap to fill in

The **essential spectral radius** of T, $r_e(T)$, is defined by

$$r_e(T) = \sup_{\lambda \in \sigma_e(T)} |\lambda| \,.$$

This is also equal to the spectral radius of $T + \mathcal{K}$ in the Calkin algebra and also of $T + \bar{\mathcal{F}}$ in $\mathcal{B}/\bar{\mathcal{F}}$.

Thus

$$r_e(T) = \inf_{n \in \mathbb{N}} \operatorname{dist}(T^n, \mathcal{K})^{1/n} = \lim_{n \to \infty} \operatorname{dist}(T^n, \mathcal{K})^{1/n}$$

$$= \inf_{n \in \mathbb{N}} \operatorname{dist}(T^n, \mathcal{F})^{1/n} = \lim_{n \to \infty} \operatorname{dist}(T^n, \mathcal{F})^{1/n} \,.$$

Gap to fill in

5 Some important classes of operators

Here we introduce several classes of operators.

Definition 5.1 Let $T \in \mathcal{B}$. We say that T is a **Riesz operator** if $\lambda I - T$ is Fredholm for all non-zero complex numbers λ.

Thus T is Riesz if and only if $r_e(T) = 0$.

We say that T is **quasicompact** if $r_e(T) < 1$.

Warning! this definition of quasicompact is the one in (the English translation of) Heuser, Page 216, Exercise 4. Some authors use quasicompact to mean power compact instead (see below).

With our definition, T is quasicompact if and only if there exists $n \in \mathbb{N}$ with $\mathrm{dist}(T^n, \mathcal{K}) < 1$.

We say that T is power compact if there exists a positive integer N such that T^N is compact.

Equivalently, this says that $T + \mathcal{K}$ is a nilpotent element of the Calkin algebra.

Certainly we have the following implications:

compact $\Longrightarrow$ power compact $\Longrightarrow$ Riesz $\Longrightarrow$ quasicompact.
In general, none of these implications can be reversed.

Gap to fill in

A useful property of a bounded linear operator T on a Banach space is that each spectral element of T which lies in the unbounded component of the complement of the essential spectrum of T is an eigenvalue of finite multiplicity. Further, if there are infinitely many of them, then they cluster only on the essential spectrum.

We will discuss the relevant theory in more detail in a future session.

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