SCHOOL OF MATHEMATICAL SCIENCES

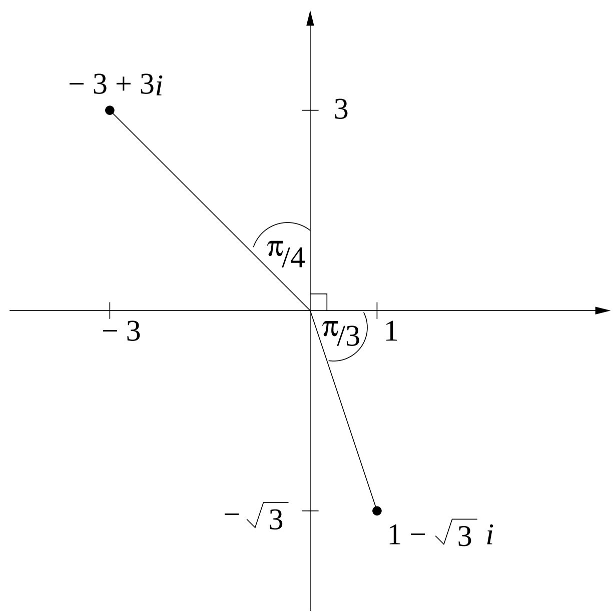
G11LMA Linear Mathematics 2002-3

Examination Solutions

1. (a) i. $(2+5i)(3-i) = (6+5)+(15-2)i = 11+13i$. Real part is 11, imaginary part is 13.

   ii. $\dfrac{2+5i}{3-i} = \dfrac{(2+5i)(3+i)}{(3-i)(3+i)} = \dfrac{1+17i}{10} = \dfrac{1}{10} + \dfrac{17}{10}i$ Real part is $\dfrac{1}{10}$, imaginary part is $\dfrac{17}{10}$.

   (b)



$$\text{Arg}\,(-3+3i) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\text{Arg}\,(1-\sqrt{3}i) = -\frac{\pi}{3}$$

$$|-3+3i| = \sqrt{3^2+3^2} = \sqrt{18} \quad \text{so}$$
$$-3+3i = \sqrt{18}e^{i(3\pi/4)}$$

$$|1-\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2 \quad \text{so}$$
$$1-\sqrt{3}i = 2e^{-i\pi/3}.$$

   (c)

$$\begin{aligned}(1-\sqrt{3}i)^9 &= (2e^{-i\frac{\pi}{3}})^9 \\ &= 2^9 e^{-i9\frac{\pi}{3}} \\ &= 2^9 e^{-i3\pi} \\ &= 512 \times (-1) = -512.\end{aligned}$$

2. (a)

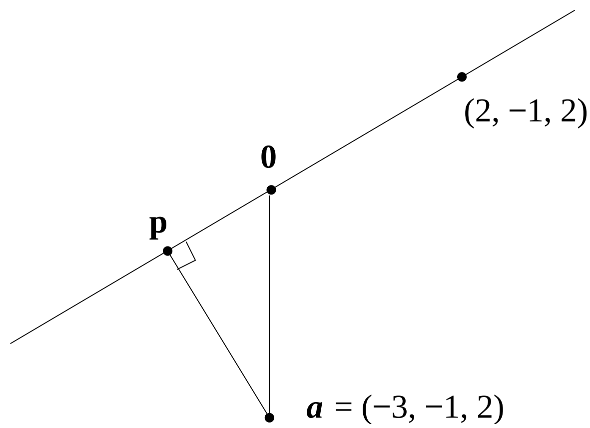
$$\|\boldsymbol{a}\| = \sqrt{2^2+1^2+4^2} = \sqrt{21}$$
$$\|\boldsymbol{b}\| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = 2\times(-1)+(-1)\times 2+4\times(-2) = -12$$
$$\boldsymbol{a}\times\boldsymbol{b} = (2-8,-4+4,4-1) = (-6,0,3).$$

$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos(\theta)$ gives us

$$\cos(\theta) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} = \frac{-12}{3\sqrt{21}} = \frac{-4}{\sqrt{21}}.$$

(b)



$\boldsymbol{p}$ is the component of $\boldsymbol{a}$ along $(2,-1,-2)$.

Set $\boldsymbol{b} = (2,-1,-2)$. So $\boldsymbol{p} = \dfrac{(\boldsymbol{a}\cdot\boldsymbol{b})}{(\boldsymbol{b}\cdot\boldsymbol{b})}\boldsymbol{b} = \dfrac{(-6+1-4)}{9}\boldsymbol{b} = -\boldsymbol{b} = (-2,1,2)$.

Distance from $\boldsymbol{a}$ to L is then

$$\|\boldsymbol{a}-\boldsymbol{p}\| = \sqrt{1^2+2^2+0} = \sqrt{5}.$$

3. (a)

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 3 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 5 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3-10 & -2-4 \\ 6+12 & -4+15 \\ -10+16 & -4+20 \end{bmatrix} = \begin{bmatrix} -7 & -6 \\ 18 & 11 \\ 6 & 16 \end{bmatrix}$$

$$\begin{bmatrix} i & 2 \\ 3 & -i \end{bmatrix}\begin{bmatrix} 1+i & -i \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} i-1+4 & 1+2 \\ 3+3i-2i & -3i-i \end{bmatrix} = \begin{bmatrix} 3+i & 3 \\ 3+i & -4i \end{bmatrix}.$$

(b) Using row operations which do not change the value of the determinant,

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 6 & -1 & 16 \\ 1 & 6 & 0 & 27 \\ 3 & 10 & -2 & 38 \end{vmatrix} = \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix} \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & 1 & 10 \\ 0 & 4 & 1 & 24 \\ 0 & 4 & 1 & 29 \end{vmatrix}$$

$$= R_4 - R_3 \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & 1 & 10 \\ 0 & 4 & 1 & 24 \\ 0 & 0 & 0 & 5 \end{vmatrix} = R_3 - 2R_2 \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & 1 & 10 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= 1 \times (2) \times (-1) \times 5 = -10$$

as the last is the determinant of a triangular matrix.

(c) We may take $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$. Then $\det(A+B) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\right) = 10$ while $\det(A) = \det(B) = 0$. Thus $\det(A+B) = \det(A) + \det(B) + 10$. as required.

4. (a) We form the augmented matrix and row reduce to echelon form.

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 3 & -6 & 11 & 2 & 9 \\ 2 & -4 & 8 & 1 & 7 \end{array}\right] \sim \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 0 & 0 & 2 & -1 & 3 \\ 0 & 0 & 2 & -1 & 3 \end{array}\right]$$

$$\sim R_3 - R_2 \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 0 & 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is now in echelon form and we can see that the equations are consistent, with leading variables x_1, x_3 and free variables x_2, x_4.

To find the general solution, set $x_2 = s, x_4 = t$. The second row of the echelon matrix gives $2x_3 - x_4 = 3$ so

$$x_3 = \frac{1}{2}(3 + x_4) = \frac{3}{2} + \frac{t}{2}.$$

The first row gives us

$$x_1 - 2x_2 + 3x_3 + x_4 = 2, \quad \text{i.e.}$$
$$x_1 - 2s + 3\Big(\frac{3}{2} + \frac{t}{2}\Big) + t = 2, \quad \text{giving}$$
$$x_1 = 2s + \Big(2 - \frac{9}{2}\Big) - \frac{5}{2}t = 2s - \frac{5}{2}t - \frac{5}{2}.$$

The general solution is thus

$$(x_1, x_2, x_3, x_4) = \Big(2s - \frac{5}{2}t - \frac{5}{2}, s, \frac{3}{2} + \frac{t}{2}, t\Big)$$
$$= \Big(-\frac{5}{2}, 0, \frac{3}{2}, 0\Big) + s(2, 1, 0, 0) + t\Big(-\frac{5}{2}, 0, \frac{1}{2}, 1\Big).$$

(b) Using Gauss-Jordan inversion

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 3 & -1 & 7 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{c} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\sim R_3 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\sim R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\sim R_1 - 2R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & -2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

So the inverse to the original matrix is

$$\begin{bmatrix} -3 & 5 & -2 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

(This may also be done using $\det$ and adj.)

The second matrix is quickly inverted using the formula for inverting 2×2 matrices. When $ad - bc \neq 0$, the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Here $ad - bc$ is $i(1-i) - i(1+i) = 2$.

Thus the required inverse matrix is

$$\frac{1}{2}\begin{bmatrix} 1-i & -i \\ -(1+i) & i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} & -\frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} & \frac{i}{2} \end{bmatrix}.$$

5. $W \subseteq V$ is a subset of V if $\boldsymbol{w}_1 + k\boldsymbol{w}_2 \in W$ for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$ and all scalars k.

$$W = \left\{ \boldsymbol{A} \in M_{n,n}(\mathbb{R}) \,\middle|\, \boldsymbol{A} = \boldsymbol{A}^\top \right\}$$

Let

$$\boldsymbol{w}_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

$$\boldsymbol{w}_1 \in W \Rightarrow a_{ij} = a_{ji}, \boldsymbol{w}_2 \in W \Rightarrow b_{ij} = b_{ji}$$

$$
\begin{aligned}
\boldsymbol{w}_1 + k\boldsymbol{w}_2 &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} + k \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} kb_{11} & kb_{12} & \dots & kb_{1n} \\ kb_{21} & kb_{22} & \dots & kb_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ kb_{n1} & kb_{n2} & \dots & kb_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} + kb_{11} & a_{12} + kb_{12} & \dots & a_{1n} + kb_{1n} \\ a_{21} + kb_{21} & a_{22} + kb_{22} & \dots & a_{2n} + kb_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} + kb_{n1} & a_{n2} + kb_{n2} & \dots & a_{nn} + kb_{nn} \end{bmatrix}
\end{aligned}
$$

If the $(i, j)^{\text{th}}$ element of this matrix is c_{ij}, then $c_{ij} = a_{ij} + kb_{ij} = a_{ji} + kb_{ji} = c_{ji}$.

$\therefore$ $\boldsymbol{w}_1 + k\boldsymbol{w}_2 \in W$ for each $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, $k \in \mathbb{R}$ and hence W is a subspace.

$$
\text{Natural basis} : \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}, \right.
$$

$$
\cdots \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},
$$

$$
\left. \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\}
$$

If $\boldsymbol{w} \in W$, $\boldsymbol{w} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \dots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{bmatrix} \quad (a_{ij} = a_{ji})$

$$\text{Then } \boldsymbol{w} = a_{12} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$+ \cdots + a_{1n} \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$+ \cdots + a_{11} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{nn} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and so this set certainly spans W.

If we form

$$k_{12} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + k_{13} \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots +$$

$$k_{n-1\,n} \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & & & & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} + k_{11} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + k_{nn} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\text{we get } \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \dots & k_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{1n} & k_{2n} & k_{3n} & \dots & k_{nn} \end{bmatrix}.$$

If this is to sum to $\mathbf{0}$, then $\begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \dots & k_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{1n} & k_{2n} & k_{3n} & \dots & k_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$

i.e. each $k_{ij} = 0$, establishing linear independence. Hence, the stated set is a basis.

The number of elements in this basis is (the number of strictly upper diagonal elements) + (number of diagonal elements) $= \frac{1}{2}(n^2 - n) + n = \frac{1}{2}(n^2 + n) = \underline{\frac{1}{2}n(n+1)}$.

6.

$$R(T) = \left\{ \boldsymbol{w} \in V \,\middle|\, \boldsymbol{w} = T(\boldsymbol{v}) \text{ for some } \boldsymbol{v} \in V \right\}$$
$$\ker(T) = \left\{ \boldsymbol{v} \in V \,\middle|\, T(\boldsymbol{v}) = \boldsymbol{0} \right\}$$

(a) For T to be a linear transformation, $T(\boldsymbol{v}_1 + k\boldsymbol{v}_2) = T(\boldsymbol{v}_1) + kT(\boldsymbol{v}_2)$ for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $k \in \mathbb{R}$.

$$\boldsymbol{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$
$$k\boldsymbol{v}_2 = k \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} kx_2 \\ ky_2 \\ kz_2 \end{bmatrix}$$

$$\therefore \quad \boldsymbol{v}_1 + k\boldsymbol{v}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} kx_2 \\ ky_2 \\ kz_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ y_1 + ky_2 \\ z_1 + kz_2 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad T(\boldsymbol{v}_1 + k\boldsymbol{v}_2) = T \begin{bmatrix} x_1 + kx_2 \\ y_1 + ky_2 \\ z_1 + kz_2 \end{bmatrix} &= \begin{bmatrix} 2(x_1 + kx_2) + (y_1 + ky_2) \\ (x_1 + kx_2) + 3(y_1 + ky_2) + (z_1 + kz_2) \\ (y_1 + ky_2) + 2(z_1 + kz_2) \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + y_1 + k(2x_2 + y_2) \\ x_1 + 3y_1 + z_1 + k(x_2 + 3y_2 + z_2) \\ y_1 + 2z_1 + k(y_2 + 2z_2) \end{bmatrix} = \begin{bmatrix} 2x_1 + y_1 \\ x_1 + 3y_1 + z_1 \\ y_1 + 2z_1 \end{bmatrix} + \begin{bmatrix} k(2x_2 + y_2) \\ k(x_2 + 3y_2 + z_2) \\ k(y_2 + 2z_2) \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + y_1 \\ x_1 + 3y_1 + z_1 \\ y_1 + 2z_1 \end{bmatrix} + k \begin{bmatrix} 2x_2 + y_2 \\ x_2 + 3y_2 + z_2 \\ y_2 + 2z_2 \end{bmatrix} = T \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + kT \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}. \end{aligned}$$

i.e. $T(\boldsymbol{v}_1 + k\boldsymbol{v}_2) = T(\boldsymbol{v}_1) + kT(\boldsymbol{v}_2)$ and hence T is a linear transformation.

Let $\boldsymbol{v} \in \ker(T)$. Then $T(\boldsymbol{v}) = \boldsymbol{0}$.

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow T(\boldsymbol{v}) = T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 3y + z \\ y + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \therefore \qquad 2x + y &= 0 && (1) \\ x + 3y + z &= 0 && (2) \\ y + 2z &= 0 && (3) \end{aligned}$$

(3)–$(1) \Rightarrow z = x$. In (2), $x + 3y + x = 0 \Rightarrow 2x + 3y = 0$ (4). (4)–$(1) \Rightarrow y = 0$. Hence, $x = z = 0$ also.

$\therefore$ If $\boldsymbol{v} \in \ker(T)$, then $\boldsymbol{v} = \mathbf{0}$ i.e. $\ker(T) = \{\mathbf{0}\}$

$$\begin{aligned} \text{rank}\,(T) &= \text{ dimension of } k(T) \\ \text{nullity}\,(T) &= \text{ dimension of } \ker(T) \end{aligned}$$

rank and nullity formula: $\text{rank}\,(T) + \text{nullity}\,(T) = \dim(V)$

In this case, $\dim(V) = \dim(\mathbb{R}^3) = 3;\quad \text{nullity}\,(T) = \dim\{\mathbf{0}\} = 0.$

$\therefore \quad \text{rank}\,(T) = 3 - 0 = 3.$

If $T(\boldsymbol{v}_1) = T(\boldsymbol{v}_2)$, then $T(\boldsymbol{v}_1) - T(\boldsymbol{v}_2) = \mathbf{0}$. But since T is a linear transformation, $T(\boldsymbol{v}_1) - T(\boldsymbol{v}_2) = T(\boldsymbol{v}_1 - \boldsymbol{v}_2)$.

$$\therefore \quad T(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \mathbf{0}, \text{ so } \boldsymbol{v}_1 - \boldsymbol{v}_2 \in \ker(T).$$

But $\ker(T) = \{\mathbf{0}\}$, so $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \mathbf{0}$ i.e. $\boldsymbol{v}_1 = \boldsymbol{v}_2$.

(b) Let $\boldsymbol{u}$ represent the function f and $\boldsymbol{v}$ represent g.

$$\begin{aligned} T(\boldsymbol{u} + k\boldsymbol{v}) = (f + kg)''(x) &= f''(x) + kg''(x) \\ &= T(\boldsymbol{u}) + kT(\boldsymbol{v}). \end{aligned}$$

$\therefore \quad T$ is a linear transformation.

$$\ker(T) = \left\{ \boldsymbol{v} \in V \,\middle|\, T(\boldsymbol{v}) = \mathbf{0} \right\}$$

i.e. if $\boldsymbol{v}$ represents the function f, then $f''(x) \equiv 0$.
i.e. $\ker(T)$ comprises all continuous, linear functions of x.

7.

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$C_1 \mapsto C_1 - C_3 : \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ -(2-\lambda) & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{i.e. } (2-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0.$$

Expand by $R1$:

$$\begin{aligned} &(2-\lambda)\Big\{ \big[(3-\lambda)(2-\lambda) - 1\big] - [0 + 1] \Big\} = 0 \\ &(2-\lambda)(\lambda^2 - 5\lambda + 4) = 0 \\ &(2-\lambda)(\lambda - 4)(\lambda - 1) = 0. \end{aligned}$$

$\therefore$ The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 4,\ \lambda_3 - 1$

$$\lambda_1 = 2 : \quad \boldsymbol{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2\alpha \\ 2\beta \\ 2\gamma \end{bmatrix}$$

$$
\begin{aligned}
2\alpha + \beta &= 2\alpha && (4)\\
\alpha + 3\beta + \gamma &= 2\beta && (5)\\
\beta + 2\gamma &= 2\gamma && (6)
\end{aligned}
$$

(4) & (6) $\Rightarrow \beta = 0$, (5) $\Rightarrow \alpha = -\gamma$

$\therefore$ An eigenvector is $\alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\alpha \neq 0$.

$$\lambda_2 = 4: \quad \boldsymbol{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 4\alpha \\ 4\beta \\ 4\gamma \end{bmatrix}$$

$$
\begin{aligned}
2\alpha + \beta &= 4\alpha && (7)\\
\alpha + 3\beta + \gamma &= 4\beta && (8)\\
\beta + 2\gamma &= 4\gamma && (9)
\end{aligned}
$$

(9) $\Rightarrow \beta = 2\gamma$, (7) $\Rightarrow \beta = 2\alpha$ so $\alpha = \gamma$, $\beta = 2\alpha$

$\therefore$ An eigenvector is $\alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\alpha \neq 0$.

$$\lambda_3 = 1: \quad \boldsymbol{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$
\begin{aligned}
2\alpha + \beta &= \alpha && (10)\\
\alpha + 3\beta + \gamma &= \beta && (11)\\
\beta + 2\gamma &= \gamma && (12)
\end{aligned}
$$

(10) $\Rightarrow \beta = -\alpha$, (12) $\Rightarrow \beta = -\gamma$, in (11) $-\beta + 3\beta - \beta = \beta$ $\quad \therefore \alpha = \gamma$, $\beta = -\alpha$

An eigenvector is $\alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\alpha \neq 0$.

Normalised eigenvectors are therefore

$$\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}; \quad \boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}; \quad \boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\boldsymbol{P} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

will be such that

$$\boldsymbol{P}^\top \boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

NB other column orders of $\boldsymbol{P}, \boldsymbol{D}$ are possible.

$$Q = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

check : $$\begin{aligned} \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 + x_3 \\ x_2 + 2x_3 \end{bmatrix} \\ &= x_1(2x_1 + x_2) + x_2(x_1 + 3x_2 + x_3) + x_3(x_2 + 2x_3) \\ &= 2x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3. \end{aligned}$$

Set $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$

$$\begin{aligned} \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} &= (\boldsymbol{P}\boldsymbol{y})^\top \boldsymbol{A}\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}^\top \boldsymbol{P}^\top \boldsymbol{A}\boldsymbol{P}\boldsymbol{y} = \boldsymbol{y}^\top \boldsymbol{D}\boldsymbol{y} \\ &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2y_1^2 + 4y_2^2 + y_3^2. \end{aligned}$$

$\therefore$ For this choice of $\boldsymbol{P}, \boldsymbol{D}$ (other choices are possible),

$$\begin{aligned} &Q = \alpha y_1^2 + \beta y_2^2 + \gamma y_3^2 \\ &\alpha = 2,\ \beta = 4,\ \gamma = 1 \end{aligned}$$

$\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x} = \boldsymbol{P}^\top \boldsymbol{x}$ (since $\boldsymbol{P}$ is orthogonal)

i.e. $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{2}}(x_1 - x_3) \\ y_2 &= \frac{1}{\sqrt{6}}(x_1 + 2x_2 + x_3) \\ y_3 &= \frac{1}{\sqrt{3}}(x_1 - x_2 + x_3). \end{aligned}$$

8.

$$k_1 \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} + k_3 \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad k_1 + ik_3 = 0 \qquad (1)$$
$$k_2 + k_3 = 0 \qquad (2)$$
$$ik_1 + ik_2 - k_3 = 0 \qquad (3)$$

$(3) \times(-i) \Rightarrow k_1 + k_2 + ik_3 = 0$. Since $k_1 + ik_3 = 0$ (from (1)), this gives $k_2 = 0$. Hence, $k_3 = 0$ (from (2)) and then $k_1 = 0$ (from $((1)$). $\quad \therefore \quad k_1 = k_2 = k_3 = 0$.

Hence, the vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are independent.

Since V is three-dimensional, any set of three independent vectors is a basis. Such a set is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, which is therefore a basis.

Define $\boldsymbol{w}_1 = \boldsymbol{v}_1$ and choose $\boldsymbol{w}_2$ such that $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \lambda \boldsymbol{v}_1$

$$\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle = \langle \boldsymbol{v}_2 - \lambda \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle = \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle - \lambda \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle.$$

$\therefore$ choosing $\lambda = \dfrac{\langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}$ will yield $\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle = 0$.

$$\langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle = \underbrace{[0, 1+i]}_{\boldsymbol{v}_2^\top} \underset{\overline{\boldsymbol{v}}_1}{\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}} = +1$$

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle = \begin{bmatrix} 1 & 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = 2$$

$\therefore \quad \lambda = \frac{1}{2}$

$$\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ i/2 \end{bmatrix}$$
$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \mu \boldsymbol{w}_2 - \nu \boldsymbol{w}_1$$
$$\langle \boldsymbol{w}_3, \boldsymbol{w}_2 \rangle = \langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle - \mu \langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle - \nu \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle$$

$\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = 0$ since $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are othogonal.

$$\therefore \quad \langle \boldsymbol{w}_3, \boldsymbol{w}_2 \rangle = \langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle - \mu \langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle.$$

$\therefore$ Choosing $\mu = \langle \boldsymbol{v}_3, \boldsymbol{w}_2\rangle / \langle \boldsymbol{w}_2, \boldsymbol{w}_2\rangle$ guarantees that $\boldsymbol{w}_3$, $\boldsymbol{w}_2$ are orthogonal.

$$\begin{aligned}
\langle \boldsymbol{v}_3, \boldsymbol{w}_2\rangle &= \begin{bmatrix} i & 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -i/2 \end{bmatrix} = -\frac{i}{2} + 1 + \frac{i}{2} = 1 \\
\langle \boldsymbol{w}_2, \boldsymbol{w}_2\rangle &= \begin{bmatrix} -1/2 & 1 & i/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1 \\ -i/2 \end{bmatrix} = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2} \\
\therefore \mu &= 1/(3/2) = 2/3 \\
\langle \boldsymbol{w}_3, \boldsymbol{w}_1\rangle &= \langle \boldsymbol{v}_3, \boldsymbol{v}_1\rangle - \lambda\langle \boldsymbol{w}_2, \boldsymbol{w}_1\rangle - \nu\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle \\
&= \langle \boldsymbol{v}_3, \boldsymbol{v}_1\rangle - \nu\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle \text{ since } \langle \boldsymbol{w}_2, \boldsymbol{w}_1\rangle = 0.
\end{aligned}$$

Choosing $\nu = \langle \boldsymbol{v}_3, \boldsymbol{v}_1\rangle / \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle$ will guarantee that $0 = \langle \boldsymbol{w}_3, \boldsymbol{w}_1\rangle$, so $\boldsymbol{w}_3$ and $\boldsymbol{w}_1$ are orthogonal.

$$\begin{aligned}
\langle \boldsymbol{v}_3, \boldsymbol{v}_1\rangle &= \begin{bmatrix} i & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = i + i = 2i \\
\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle &= \begin{bmatrix} 1 & 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = 1 + 1 = 2 \\
\therefore \quad \nu &= \frac{2i}{2} = i.
\end{aligned}$$

Then

$$\boldsymbol{w}_3 = \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -\frac{1}{2} \\ 1 \\ i/2 \end{bmatrix} - i\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -i/3 \end{bmatrix}.$$