G11LMA Linear Mathematics

Solutions to assessed coursework 1

20 marks available, 4 marks for each question

1.(a) We can use a standard formula for $1/z$ from the notes.

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{2-5i}{29} = \frac{2}{29} - \frac{5}{29}i.$$

This gives us $\mathrm{Re}(1/z) = 2/29$ and $\mathrm{Im}(1/z) = -5/29$ (and NOT $-(5/29)i$).

Similar calculations give us $1/w = (3/25)+(4/25)i$, so $\mathrm{Re}(1/w) = 3/25$ and $\mathrm{Im}(1/w) = 4/25$.

Finally, $z/w = z(1/w) = (1/25)(2+5i)(3+4i) = (-14/25) + (23/25)i$ so $\mathrm{Re}(z/w) = -14/25$ and $\mathrm{Im}(z/w) = 23/25$.

(b) This equation has the form $az^2 + bz + c = 0$ where $a = 1$, $b = 2i$ and $c = -1 + 2i$. We can solve this using the quadratic formula. First note that $b^2 - 4ac = -4 - (-4 + 8i) = -8i = 4(-2i)$. One of the solutions to $w^2 = b^2 - 4ac$ is (by inspection or using the Argand diagram) $w = 2(1 - i)$. The solutions to the quadratic are then $z = \frac{-b\pm w}{2a}$ i.e. $z = \frac{-2i\pm 2(1-i)}{2}$, giving $z = 1 - 2i$ or $z = -1$.

If you spot that -1 is a solution, then another method is to factorise: $z^2+2iz-1+2i = (z + 1)(z - 1 + 2i)$, from which the result follows immediately.

2. We have $\|\mathbf{a}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$, $\|\mathbf{b}\| = \sqrt{(-3)^2 + 1^2 + 4^2} = \sqrt{26}$, $\mathbf{a}\cdot\mathbf{b} = -6 - 1 + 4 = -3$ and $\mathbf{a}\times\mathbf{b} = (-4 - 1, -3 - 8, 2 - 3) = (-5, -11, -1)$.

When the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, we have

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

so

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = -3/(\sqrt{6}\sqrt{26})$$

(or $-\sqrt{39}/26$).

3. Set $\mathbf{b} = (2, 1, -3)$ (a direction vector for the line). By a standard result from the notes, since the line passes through the origin, the closest point, $\mathbf{p}$, of the line to $\mathbf{a}$ is equal to the component of $\mathbf{a}$ along $\mathbf{b}$, i.e $\mathbf{p} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$. Here, $\mathbf{a} \cdot \mathbf{b} = 1$ and $\|\mathbf{b}\|^2 = 14$ so $\mathbf{p} = (1/14)\mathbf{b} = (1/14)(2, 1, -3)$. Thus the distance from $\mathbf{a}$ to L is simply $\|\mathbf{a} - \mathbf{p}\| = \|(4, -1, 2) - (1/14)(2, 1, -3)\|$ which may be calculated directly or by Pythagoras and comes out to be $\frac{\sqrt{4102}}{14}$ or $\sqrt{293/14}$.

4. There are many correct proofs.

Method I: Geometric reasoning. Let $\mathbf{n}$ be a (non-zero) normal vector to the plane M. If either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{c} \times \mathbf{d}$ are the zero vector then the result is clear. Otherwise both are normal to the plane and so must be multiples of $\mathbf{n}$, say $\mathbf{a} \times \mathbf{b} = s\mathbf{n}$ and $\mathbf{c} \times \mathbf{d} = t\mathbf{n}$ for scalars s and t. This gives $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (s\mathbf{n}) \times (t\mathbf{n}) = \mathbf{0}$, as required.

Method II: Standard formulae. We know (Rule 2.36) that whenever three vectors lie in the same plane through the origin, their scalar triple product is 0. We also have the standard formulae for vector triple products (Rule 2.34). For example, we know that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{x} = (\mathbf{x} \cdot \mathbf{a})\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\mathbf{a}.$$

Setting $\mathbf{x} = \mathbf{c} \times \mathbf{d}$ we obtain

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = ((\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a})\mathbf{b} - ((\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b})\mathbf{a}.$$

Since both of the scalar triple products $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}$ and $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}$ are 0, the result is the zero vector, as required.

5. You can multiply two matrices if the first has as many columns as the second has rows. The resulting matrix has as many rows as the first matrix and as many columns as the second matrix. The products which are defined are AA (or A^2), BB (or B^2), CB and AC. Of these products, B^2, CB and AC have 2 columns. We find these three products are

$$B^2 = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}, \quad CB = \begin{bmatrix} 6 & -8 \\ 3 & -4 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad AC = \begin{bmatrix} -1 & -2 \\ -2 & 1 \\ -3 & 4 \end{bmatrix}.$$