5 Eigenvalues and Eigenvectors

These notes are heavily based on the Spring Semester 2002-3 lecture notes of R.H. Tew, with very minor modifications by J.F. Feinstein 2003-4. Note that Dr Tew uses bold letters for matrices, and that he usually distinguishes between points and their position vectors.

5.1 Motivation

• Systems of ordinary differential equations

Consider the system of ordinary differential equations

$$\dot{x} = -2x + y$$

$$\dot{y} = x - 2y,$$

where $\dot{x} = \dfrac{dx(t)}{dt}$, $\dot{y} = \dfrac{dy(t)}{dt}$. One way of proceeding is to eliminate one of the variables ($y(t)$ say) by differentiating the first equation with respect to t and then using both equations to eliminate $\dot{y}$ and y, respectively, leaving a second order ordinary differential equation for $x(t)$ alone:

$$\ddot{x} + 4\dot{x} + 3x = 0.$$

Seeking an exponential solution $x(t) = Ae^{\lambda t}$ quickly leads to the auxiliary equation

$$\lambda^2 + 4\lambda + 3 = 0$$

for λ, from which $\lambda = -1, -3$ follows. The general solution for $x(t)$ is then

$$x(t) = Ae^{-t} + Be^{-3t},$$

where A and B are arbitrary constants, and $y(t)$ can be obtained by substituting this solution back into the first equation.

Another approach is to assume an exponential solution

$$x(t) = \alpha e^{\lambda t}, \; y(t) = \beta e^{\lambda t}$$

directly. Then we quickly find that

$$\lambda\alpha = -2\alpha + \beta$$

$$\lambda\beta = \alpha - 2\beta$$

follows, since the exponential terms $e^{\lambda t}$ cancel throughout. If we write

$$\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \; \boldsymbol{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

then these equations for α and β can be re-expressed in the matrix form

$$\boldsymbol{A}\,\boldsymbol{v} \;=\; \lambda \boldsymbol{v}.$$

Since $\lambda \boldsymbol{v} \;=\; \lambda \boldsymbol{I}\,\boldsymbol{v}$, where $\boldsymbol{I}$ is the 2×2 identity matrix, we can re-write this as

$$\boldsymbol{B}\,\boldsymbol{v} \;=\; \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where the 2×2 matrix $\boldsymbol{B}$ is given by $\boldsymbol{B} \;=\; \boldsymbol{A} - \lambda \boldsymbol{I}$.

We are not interested in solutions for which $\boldsymbol{v} \;=\; \boldsymbol{0}$, since then $\alpha \;=\; \beta \;=\; 0$, which would then imply $x(t) \;=\; 0,\; y(t) \;=\; 0$, which are obviously solutions. Hence, we must have

$$\det\boldsymbol{B} \;=\; \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \;=\; 0.$$

This is because if $\det \boldsymbol{B} \neq \boldsymbol{0}$, then $\boldsymbol{B}^{-1}$ exists and $\boldsymbol{B}\,\boldsymbol{v} \;=\; \boldsymbol{0}$ implies $\boldsymbol{B}^{-1}\,\boldsymbol{B}\,\boldsymbol{v} \;=\; \boldsymbol{B}^{-1}\,\boldsymbol{0} \;=\; \boldsymbol{0}$ i.e. $\boldsymbol{v} \;=\; \boldsymbol{0}$, which we have discounted as a possibility.

In this case,

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

and so

$$\begin{aligned} \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) &= (-2\; - \lambda)\,(-2 - \lambda) - 1 \\ &= \lambda^2 + 4\lambda + 3. \end{aligned}$$

Hence λ satisfies the quadratic equation

$$\lambda^2 + 4\lambda + 3 = 0,$$

and the solution for $x(t)$ (and then $y(t)$) follows exactly as it did previously.

Hence, the matrix equation $\boldsymbol{A}\,\boldsymbol{v} = \lambda \boldsymbol{v}$ led us directly to the solution of the differential equations that we wished to solve.

- **Coordinate transformations**

Consider the change of coordinates

$$x' \;=\; ax \;+\; by$$

$$y' \;=\; cx \;+\; dy,$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\boldsymbol{A} \;=\; \begin{bmatrix} a\;b \\ c\;d \end{bmatrix}$. This change of coordinates might be a rotation of axes, for example, and the point with position vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is then mapped to the new point $\begin{bmatrix} x' \\ y' \end{bmatrix}$.

A natural question to ask is: which straight lines through the origin in the $x - y$ plane are preserved (i.e. left unchanged) by this transformation? It might (and generally will) be the case that each point on the line (other than the origin) is mapped to a different point on the line, but the line as a whole remains unchanged (assuming, of course, that such a line exists).

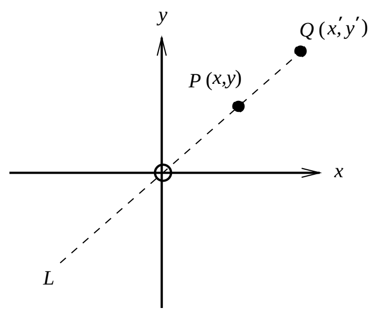

Figure 2.1

Suppose that the point P (not the origin) with position vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is mapped to the point Q with position vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$. If they are both to lie on the same straight line L through the origin, then their position vectors must be parallel. That is, there exists some scalar λ such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Setting $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ quickly leads to the matrix equation $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, which is identical in form to the matrix equation that we obtained at the end of the previous example.

Both of these motivating examples extend to higher-dimensional cases, when $\boldsymbol{A}$ will be an $n \times n$ matrix, and $\boldsymbol{v}$ an $n \times 1$ column vector.

5.2 Definition

These examples lead us to the following definition:

Definition

The $n \times 1$ column vector $\boldsymbol{v}$ is an **eigenvector** of the $n \times n$ matrix $\boldsymbol{A}$, with corresponding (or associated) **eigenvalue** λ, if $\boldsymbol{v} \neq \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.

Remarks

1. $\boldsymbol{v}$ must be non-zero. The zero column vector is **never** an eigenvector.
2. The corresponding eigenvalue λ is a scalar.
3. λ and $\boldsymbol{v}$ are in partnership.
4. if $\boldsymbol{v}$ is an eigenvector with eigenvalue λ, then so is any non-zero scalar multiple of it.

5.3 Calculating Eigenvalues - The Characteristic Polynomial and the Characteristic Equation

Since $\boldsymbol{A}\,\boldsymbol{v} = \lambda\boldsymbol{v}$, we know that $(\boldsymbol{A} - \lambda\boldsymbol{I})\,\boldsymbol{v} = \boldsymbol{0}$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Suppose $(\boldsymbol{A} - \lambda\boldsymbol{I})^{-1}$ exists. Then $\boldsymbol{v} = (\boldsymbol{A} - \lambda\boldsymbol{I})^{-1}\boldsymbol{0} = \boldsymbol{0}$. But this is not allowed, and so we must insist that $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular, i.e. $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$.

Hence, the eigenvalue λ satisfies the equation

$$\det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0.$$

Remarks

1. If $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then $\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$.

2. $\det(\boldsymbol{A} - \lambda\boldsymbol{I})$ is sometimes written $|\boldsymbol{A} - \lambda\boldsymbol{I}|$.

3. By expanding $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$ by whichever means you like, we see that the right-hand side of the equation satisfied by λ is a **polynomial in $\boldsymbol{\lambda}$ of degree $\boldsymbol{n}$**. It is called the **CHARACTERISTIC POLYNOMIAL**. The equation itself is called the **CHARACTERISTIC EQUATION**. Thus, $\boldsymbol{A}$ generally has n eigenvalues, possibly complex, each with their own associated eigenvectors. The case of repeated roots will be mentioned later.

4. For each possible value of λ, the corresponding eigenvectors are calculated by considering the system of equations $\boldsymbol{A}\,\boldsymbol{v} = \lambda\,\boldsymbol{v}$.

5.4 Examples

- **Distinct roots to the characteristic polynomial**

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \quad \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 1 & 0 & -3-\lambda \end{vmatrix}.$$

Expanding by column 1:

$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = (1-\lambda)\,[(1-\lambda)\,(-3-\lambda) - 0] - 0\;[2(-3-\lambda) - 0] + 1.\,[4 - (1-\lambda)]$

$= (1-\lambda)\,(\lambda-1)\,(\lambda+3) + (\lambda+3)$

$= (\lambda+3)\;[(1-\lambda)\,(\lambda-1) + 1\,]$

$= (\lambda+3)\,(-\lambda^2 + 2\lambda)$

$= -\lambda\,(\lambda+3)\,(\lambda-2).$

Hence $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = -\lambda(\lambda+3)(\lambda-2)$ and so the eigenvalues of $\boldsymbol{A}$ are $0,\ -3$ and 2.

Remarks

1. It *is* possible to have a zero eigenvalue; it is *not* possible to have a zero eigenvector.

2. We are lucky with the factorisation in this example. For a 3×3 matrix, the characteristic polynomial will always be a cubic. If you cannot factorise it directly, try some small values (eg $\pm 1,\ \pm 2$) to try to spot one of the roots. This will then give you one of the factors of the cubic, allowing you to determine the remaining quadratic factor (by observation or long division, perhaps).

   To find the corresponding eigenvectors, we consider each eigenvector in turn:

1. $\boldsymbol{\lambda} = \mathbf{0}$

   Let the eigenvector be $\boldsymbol{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. Then in this case, $\boldsymbol{A}\,\boldsymbol{v} = \lambda \boldsymbol{v} = 0\,\boldsymbol{v} = \mathbf{0}$. Hence

   $$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   This leads to the system of equations

   $$\begin{array}{ccccccc} \alpha & + & 2\beta & + & \gamma & = & 0 \\ & & \beta & + & 2\gamma & = & 0 \\ \alpha & & & - & 3\gamma & = & 0 \end{array}.$$

   Thus, $\alpha = 3\gamma$ and $\beta = -2\gamma$ and $\boldsymbol{v} = \begin{bmatrix} 3\gamma \\ -2\gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\gamma \neq 0$. Hence, any non-zero multiple of $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue zero.

2. $\boldsymbol{\lambda} = \mathbf{-3}$

   This time, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = -3 \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -3\alpha \\ -3\beta \\ -3\gamma \end{bmatrix}$.

   Hence,

   $$\begin{array}{ccccccc} \alpha & + & 2\beta & + & \gamma & = & -3\alpha \\ & & \beta & + & 2\gamma & = & -3\beta \\ \alpha & & & - & 3\gamma & = & -3\gamma \end{array}.$$

   The last of these equations gives $\alpha = 0$. The first two both reduce to the same equation $2\beta + \gamma = 0$.

   Hence, $\alpha = 0,\ \gamma = -2\beta$ and so the eigenvector is $\begin{bmatrix} 0 \\ \beta \\ -2\beta \end{bmatrix} = \beta \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, $\beta \neq 0$.

   i.e. any non-zero multiple of $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue -3.

3. $\boldsymbol{\lambda = 2}$

   It is slightly more efficient to look directly at $\boldsymbol{A} - \lambda\boldsymbol{I}$, and solve the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$. (As we saw earlier, this is the same as solving $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.)

   Here this gives us

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

$$\begin{array}{rcrcrcl} -\alpha & + & 2\beta & + & \gamma & = & 0 \\ & & -\beta & + & 2\gamma & = & 0 \\ \alpha & & & - & 5\gamma & = & 0 \end{array}.$$

   The last two equations imply $\alpha = 5\gamma$, $\beta = 2\gamma$; the first is then satisfied automatically.

   Hence, in this case, the eigenvector is $\begin{bmatrix} 5\gamma \\ 2\gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$. i.e. any non-zero multiple of $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$.

Remark

Eigenvectors are never unique: you can multiply an eigenvector by any non-zero scalar and obtain another eigenvector with the same eigenvalue.

- **Repeated roots to the characteristic polynomial**

$$\boldsymbol{A} = \begin{bmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{bmatrix}$$

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 17-\lambda & -10 & -5 \\ 45 & -28-\lambda & -15 \\ 30 & 20 & 12-\lambda \end{vmatrix} = 0.$$

  Perform the following operations:

- row 2 $\mapsto$ row 2 - 3 row 1:

$$\begin{vmatrix} 17-\lambda & -10 & -5 \\ -6+3\lambda & 2-\lambda & 0 \\ -30 & 20 & 12-\lambda \end{vmatrix} = 0$$

- row 3 $\mapsto$ row 3 + 2 row 1:

$$\begin{vmatrix} 17-\lambda & -10 & -5 \\ -3(2-\lambda) & -2-\lambda & 0 \\ 4-2\lambda & 0 & 2-\lambda \end{vmatrix} = 0$$

- remove a factor $(2-\lambda)$ from row 2 and from row 3:

$$(2-\lambda)^2 \begin{vmatrix} 17-\lambda & -10 & -5 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

- expand by row 3:

$$(2-\lambda)^2\left\{2\begin{vmatrix} -10 & -5 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -17-\lambda & -10 \\ -3 & 1 \end{vmatrix}\right\} = 0,$$

  which simplifies to $(2-\lambda)^2(3+\lambda) \ = 0$.

Hence, the eigenvalues are 2 (repeated) and -3.

Remark

Notice how row (and column) operations have factorised the characteristic polynomial; this is not always possible.

• $\boldsymbol{\lambda = -3}$

In the usual notation, we have

$$\begin{bmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = -3\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -3\alpha \\ -3\beta \\ -3\gamma \end{bmatrix}$$

$$\begin{array}{rcrcrcl} 17\alpha & - & 10\beta & - & 5\gamma & = & -3\alpha \\ 45\alpha & - & 28\beta & - & 15\gamma & = & -3\beta \\ -30\alpha & + & 20\beta & + & 12\gamma & = & -3\gamma \end{array}$$

i.e.

$$\begin{array}{rcrcrcl} 4\alpha & - & 2\beta & - & \gamma & = & 0 \\ 9\alpha & - & 5\beta & - & 3\gamma & = & 0\ . \\ -6\alpha & + & 4\beta & + & 3\gamma & = & 0 \end{array}$$

Adding the second and third equations gives $\beta = 3\alpha$. In the second, this gives $\gamma = -2\alpha$, and the first is then identically satisfied.

So the eigenvector is $\begin{bmatrix} \alpha \\ 3\alpha \\ -2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$;

i.e. any non-zero multiple of $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ is an eigenvector with eigenvalue-3.

• $\boldsymbol{\lambda = 2}$

$$\begin{bmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 2\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2\alpha \\ 2\beta \\ 2\gamma \end{bmatrix}$$

Then

$$\begin{array}{rcrcrcl} 15\alpha & - & 10\beta & - & 5\gamma & = & 0 \\ 45\alpha & - & 30\beta & - & 15\gamma & = & 0 \\ -30\alpha & + & 20\beta & + & 10\gamma & = & 0 \end{array}$$

i.e.

$$\begin{array}{rcrcrcl} 3\alpha & - & 2\beta & - & \gamma & = & 0 \\ 3\alpha & - & 2\beta & - & \gamma & = & 0 \\ -3\alpha & + & 2\beta & + & \gamma & = & 0 \end{array}.$$

In this example, all three equations are identical, and so we only have one relationship between α, β and γ: $3\alpha - 2\beta - \gamma \ = \ 0$.

We therefore see that $\begin{bmatrix} \alpha \\ \beta \\ 3\alpha - 2\beta \end{bmatrix}$ is an eigenvector, provided α and β are not both zero.

If we choose $\alpha \ = \ 0$, then $\boldsymbol{v_1} = \begin{bmatrix} 0 \\ \beta \\ -2\beta \end{bmatrix} = \beta \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ is an eigenvector (for $\beta \ \neq \ 0$).

If we choose $\beta \ = \ 0$, then $\boldsymbol{v_2} = \begin{bmatrix} \alpha \\ 0 \\ 3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ is also an eigenvector (for $\alpha \neq 0$).
Furthermore, they do not depend upon one another (note the positions of the zero components) and so we have generated two independent eigenvectors corresponding to the eigenvalue $\lambda \ = \ 2$.

This is not always the case; sometimes we can only find one eigenvector for a repeated eigenvalue.

- **Complex roots to the characteristic polynomial**

$$\boldsymbol{A} \ = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix}$$

so the characteristic equation is

$$(3-\lambda)^2 \ + \ 16 \ = \ 0$$

with complex-valued eigenvalue roots $\lambda \ = \ 3 \pm 4i$. For the case $\lambda \ = \ 3 \ + \ 4i$, let the eigenvector be $\boldsymbol{v} \ = \ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Then

$$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (3 \ + \ 4i) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (3+4i)\alpha \\ (3+4i)\beta \end{bmatrix}.$$

Hence,

$$\begin{array}{rcl} 3\alpha - 4\beta & = & (3+4i)\alpha \\ 4\alpha + 3\beta & = & (3+4i)\beta \end{array}$$

or, equivalently,

$$\begin{array}{rcrcl} i\alpha & + & \beta & = & 0 \\ \alpha & - & i\beta & = & 0 \end{array}.$$

Notice that these two equations are equivalent to one another. Also, it is tempting to try to 'separate real and imaginary parts' and conclude that $\alpha = \beta = 0$. This is not so, since both α and β will themselves be complex.

Hence, we see that $\alpha = i\beta$ and so any vector (possibly complex) of the form $\beta \begin{bmatrix} i \\ 1 \end{bmatrix}, \beta \neq 0$, will be an eigenvector with eigenvalue $3 + 4i$

In this case, we will check the result directly. Choosing $\beta = 1$,
$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 3i - 4 \\ 4i + 3 \end{bmatrix} = \begin{bmatrix} i(4i + 3) \\ (4i + 3) \end{bmatrix} = (3+4i) \begin{bmatrix} i \\ 1 \end{bmatrix}$, as required.

5.5 Some Special Cases

- **Eigenvalues of diagonal matrices**

The matrix $\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is a diagonal matrix. Then

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} a_{11} - \lambda & 0 & \cdots & 0 \\ 0 & a_{22} - \lambda & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{nn} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & 0 & \cdots & 0 \\ 0 & a_{33} - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix}$$

(expanding by row 1)

$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda),$$

by continually expanding by row 1.

Hence the eigenvalues satisfy

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0.$$

i.e. the eigenvalues of a diagonal matrix are precisely the diagonal entries of the matrix.

In fact, this is a special case of the following rule.

- **Eigenvalues of triangular matrices**

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \text{ then}$$

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22}-\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

(continually expanding by row 1). i.e. the eigenvalues of a triangular (upper or lower) matrix are the diagonal entries of the matrix.

• Eigenvalues of powers of matrices

Suppose that the $n \times n$ matrix $\boldsymbol{A}$ has eigenvectors $\lambda_1, \lambda_2, \cdots, \lambda_n$, with corresponding eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}$. Consider $\boldsymbol{A}^k (= \underbrace{\boldsymbol{A}\ \boldsymbol{A}\ \boldsymbol{A} \cdots\ \boldsymbol{A}}_{k \text{ times}})$. Then

$$\boldsymbol{A}^k\ \boldsymbol{v}_1 = \boldsymbol{A}^{k-1}(\boldsymbol{A}\ \boldsymbol{v}_1) = \boldsymbol{A}^{k-1}(\lambda_1 \boldsymbol{v}_1) = \lambda_1 \boldsymbol{A}^{k-1}\ \boldsymbol{v}_1.$$

Repeating this procedure, we see that

$$\boldsymbol{A}^k\ \boldsymbol{v}_1\ =\ \lambda_1^k\ \boldsymbol{v}_1$$

i.e. λ_1^k is an eigenvalue of $\boldsymbol{A}^k$, with corresponding eigenvector $\boldsymbol{v}_1$. Obvious extensions apply to $\lambda_2^k, \cdots,\ \lambda_n^k$.