G11LMA: Linear Mathematics 2003-4
Semester 1, blow-by-blow account of the module

Lecture 1: Brief discussion of $\mathbb{R}$: points on a line. Examples (without proofs) of real numbers of various kinds. The set of ordered pairs of real numbers, $\mathbb{R}^2$. The set of complex numbers, $\mathbb{C}$, and the complex number i. The Argand diagram and the complex plane.

Lecture 2: Addition and subtraction of complex numbers. Multiplication and division of complex numbers. Conjugates and modulus.

Lecture 3: The triangle inequality. Argument and polar form. Principal argument.

Lecture 4: Exponential form. Geometrical interpretation of multiplication. Powers of complex numbers and de Moivre's Theorem. Application of de Moivre to finding $\cos(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Lecture 5: Complex-valued functions, especially polynomials. Equations in $\mathbb{C}$. Roots of general complex numbers. Roots of unity.

Lecture 6: Roots of quadratic equations. Statement (without proof) of the Fundamental Theorem of Algebra. Revision of Cartesian coordinates: $\mathbb{R}^2$ (recalled) and $\mathbb{R}^3$. Brief discussion of $\mathbb{R}^4$ and other higher-dimensional spaces. Vectors and scalars. Equality of vectors. Addition, subtraction and multiplication (by scalars) of vectors: geometrically and by coordinatewise operations. The zero vector. Distance between points and norm (length, magnitude, modulus) of vectors. Unit vectors and normalization of vectors.

Lecture 7: The parallelogram law. Scalar (dot) product with properties. Perpendicularity. Components of vectors along other (non-zero) vectors. Equations of lines in various forms.

Lecture 8: Planes and normal vectors. Components of a vector along a vector (revised) and perpendicular (orthogonal) to a vector. Orthogonal projections onto lines. Distance of a point from a line.

Lecture 9: Projections onto planes. Distance from a point to a plane.

Lecture 10: Angles between lines and planes. Vector (cross) product with properties.

Lecture 11: Vector and scalar triple products with properties. Applications. Vectors in $\mathbb{R}^n$ (real coordinates and real scalars) and $\mathbb{C}^n$ (complex coordinates and complex scalars). Definition of matrices over $\mathbb{R}$ and $\mathbb{C}$. Motivation: systems of equations, geometric transformations.

Lecture 12: Brief discussion of changing co-ordinates (new axes). Matrix notation. Equality of matrices. Rows and columns. Row and column vectors. The $ij-$entry of a matrix (the entry in row i and column j). Size (order) of a matrix. Matrix addition. Scalar multiplication. The zero matrix. Multiplication of column vectors in $\mathbb{R}^2$ by 2×2 matrices.

Lecture 13: Transformations of $\mathbb{R}^2$, including rotation through angle θ. Multiplication of general column vectors by matrices. Motivation for matrix multiplication in terms of performing two transformations, one after the other. Multiplication of matrices.

Lecture 14: Further examples and rules for matrix multiplication. The transpose of a matrix. Square matrices. Examples to show that AB may be 0 without A or B being 0 and to show that AB and BA may be different. The identity matrix.

Lecture 15: Matrix inverses and their properties. Invertible (non-singular) and singular matrices. Determinants $(ad - bc)$ of 2×2 matrices. Formula for inverse of invertible 2×2 matrices. Conjugate and Hermitian adjoint of a matrix. Special types of matrices: symmetric, Hermitian, orthogonal, unitary, triangular, diagonal and scalar matrices.

Lecture 16: Determinants: motivations (conditions for invertibility, volume/area scaling factor). Adjugates of 2×2 matrices. Determinants of 3×3 matrices. Connection with volumes.

Lecture 17: Connections with the vector cross product and the scalar triple product. Minor matrices. Expansion by any row or column. The matrix of cofactors. Determinants of $n \times n$ matrices. Adjugates (adjoints) of $n \times n$ matrices. Formula for the inverse of an $n \times n$ matrix in terms of Adj and $\det$. Singular matrices and non-singular (invertible) matrices.

Lecture 18: Standard properties of determinants stated, including an introduction to row and column operations. A quick way to calculate some 4×4 determinants. Examples of easy systems of linear equations. Connections with geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$ and with matrix multiplication.

Lecture 19: Systems of linear equations in matrix form, coefficient and augmented matrix. Gaussian elimination in the matrix context. Elementary row operations. Row equivalence of matrices. (Row) echelon form. Leading entries, pivot columns, leading variables and free variables. Solution sets.

Lecture 20: Consistent and inconsistent systems, General solutions. Reduced echelon form. Homogeneous systems of equations. The More Unknowns Theorem. Square coefficient matrices.

Lecture 21: Solving systems using Cramer's rule. Gauss-Jordan inversion: method and examples. Standard equivalent conditions in terms of matrices for systems of n equations in n unknowns to have a **unique** solution (combining our earlier results): row equivalence of A to I, invertibility of A, determinant of A non-zero, no non-trivial solutions to $A\mathbf{x} = \mathbf{0}$, for all $\mathbf{b}$ there is **at least** one solution to $A\mathbf{x} = \mathbf{b}$. Similar standard equivalent conditions in terms of matrices for systems of n equations in n unknowns not to have a unique solution.

Lecture 22: Eigenvalues and eigenvectors, characteristic polynomial and characteristic equation for square matrices. Applications.

END OF SEMESTER 1