G11LMA: Linear Mathematics, Problems on Autumn Semester material

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These problems are primarily for the the weekly workshops and tutorials supporting the Core modules G11ACF, G11CAL, G11LMA.
There is one set of problems for each chapter and one set of (mostly rather hard) prize problems. Some of the problems are intended to be A-level revision, others (module problems) are of variable levels of difficulty.
Many of the non-prize problems may be solved using MAPLE. This is a good MAPLE exercise and a good way to check your answers. You should also, however, be able to solve most of these problems without using MAPLE.
Solutions to the non-prize problems will be made available from the module web page in due course.

1 Complex Numbers

A-level revision

Problem 1.1 Illustrate on a diagram the following points of $\mathbb{R}^2$: (i) $(2, -1)$ (ii) $(-2.5, 1.5)$ (iii) $(-\sqrt{2}, -\sqrt{2})$, (iv) $(2, \mathrm{e})$ (v) $(-1, 0)$ (vi) $(-3, 0)$.

Module problems

Problem 1.2 Using the Argand diagram, illustrate the following complex numbers:
(i) -1, (ii) $2i$, (iii) $2 - 3i$, (iv) 3, (v) $-2 + 2i$, (vi) $1 + 2i$, (vii) $-3 - 2i$.

Problem 1.3 Let $z = 3 - 2i$ and $w = 1 + 4i$.

(a) Find $z + w$, $z - w$, z^2 and zw.

(b) Find $\bar{z}$ and $|z|$. Verify that the formula $z\bar{z} = |z|^2$ is true in this case.

(c) Find the real and imaginary parts of the following numbers: (i) $1/z$, (ii) $1/w$, (iii) z/w, (iv) w/z.

Problem 1.4 (a) By plotting them on the Argand diagram, find the **principal** argument (Arg) of each of the following complex numbers, and express each of them in polar form and in exponential form:

(i) $-1-i$, (ii) $3i$, (iii) $-1+\sqrt{3}i$, (iv) $\sqrt{3}-i$, (v) $2+2i$.

(b) Use your answer to (a)(iii) and the rule given in the notes for taking powers of non-zero complex numbers to find $(-1+\sqrt{3}i)^7$.

(c) Find **all** possible arguments for the complex number $\sqrt{3}-i$.

Problem 1.5 Find all the 6th roots of -64: (i) in exponential form; (ii) in the form $x+yi$.

Problem 1.6 Find the two complex solutions to the equation

$$z^2+(3+3i)z+4i=0.$$

[Hint: first find the two complex square roots of $2i$.]

Problem 1.7 Explain why, if $x<0$, $\arctan(y/x)$ is never equal to $\operatorname{Arg}(x+iy)$.

Problem 1.8 (a) Give an example to show that for complex numbers z and w it is not always true that $\operatorname{Re}(zw)=\operatorname{Re}(z)\operatorname{Re}(w)$.

(b) For which pairs of complex numbers z and w is it true that $\operatorname{Re}(zw)=\operatorname{Re}(z)\operatorname{Re}(w)$?

(c) Let $z=1+i$. For which complex numbers w, if any, do we have $\operatorname{Im}(zw)=\operatorname{Im}(z)\operatorname{Im}(w)$?

(d) Repeat part (c) for the case $z=i$.

(e) Repeat part (c) in the case where $z=a+bi$ is a general complex number, giving your answer in terms of the real numbers a and b.

Problem 1.9 Given a complex number $z=x+iy$, we already know how to define $\exp(x)=\mathrm{e}^x$ (a real number) and $\exp(iy)=\mathrm{e}^{iy}$ (a complex number of modulus 1). It is standard to define

$$\exp(z)=\exp(x)\exp(iy)=\mathrm{e}^x\mathrm{e}^{iy}.$$

(a) Let z be a complex number. Show that $|\exp(z)|=\exp(\operatorname{Re}(z))$.

(b) Now suppose that z and w are complex numbers with $\exp(z)=w$. (In this case we say that z is a **complex logarithm** of w.) Show that $w\neq 0$, $\operatorname{Re}(z)=\log(|w|)$ (remember that we always use the natural logarithm, base e) and that $\operatorname{Im}(z)$ is an argument of w.

(c) Find all possible complex logarithms of $1+i$ (as defined in (b)).

(d) Show that every non-zero complex number has infinitely many different complex logarithms. How are these different complex logarithms related?

MAPLE exercise

Use MAPLE to check your answers to as many of the above problems as possible.

2 Vector Algebra and Geometry

A-level revision

Problem 2.1 Consider the vectors $\mathbf{a} = (3, -2)$ and $\mathbf{b} = (1, 2)$ in $\mathbb{R}^2$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{a}$, $2\mathbf{b}$ and $3\mathbf{a} - 2\mathbf{b}$.

Problem 2.2 Consider the vectors $\mathbf{a} = (-1, 2, 2)$ and $\mathbf{b} = (3, -4, 7)$ in $\mathbb{R}^3$. Find $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$.

Problem 2.3 (a) Find the distance from $(3, -2)$ to $(-1, 3)$ in $\mathbb{R}^2$.

(b) Find the distance from $(2, -1, 3)$ to $(-1, 3, 4)$ in $\mathbb{R}^3$.

(c) Find the following magnitudes (norms) of vectors: (i) $\|(-2, 5)\|$, (ii) $\|(-3, 2, -1)\|$.

Problem 2.4 Consider the vectors $\mathbf{a} = (-5, 12)$, $\mathbf{b} = (4, 5)$ (in $\mathbb{R}^2$) and $\mathbf{c} = (1, -2, 4)$ (in $\mathbb{R}^3$). Find unit vectors $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ in the directions of, respectively, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Problem 2.5 For the following vectors $\mathbf{a}$ and $\mathbf{b}$, calculate $\|\mathbf{a}\|$, $\|\mathbf{b}\|$ and $\mathbf{a} \cdot \mathbf{b}$. Hence find $\cos\theta$, where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, and determine whether or not $\mathbf{a}$ is perpendicular to $\mathbf{b}$: (i) $\mathbf{a} = (2, -1)$, $\mathbf{b} = (2, 2)$; (ii) $\mathbf{a} = (2, 3)$, $\mathbf{b} = (-3, 2)$; (iii) $\mathbf{a} = (-1, 3, 2)$, $\mathbf{b} = (4, 2, -1)$; (iv) $\mathbf{a} = (2, 1, -1)$, $\mathbf{b} = (3, 1, 2)$.

Problem 2.6 Find real numbers m and c such that the equation $y = mx + c$ defines the line passing through the points $(3, 4)$ and $(-2, 1)$ in $\mathbb{R}^2$.

Problem 2.7 Let $\mathbf{a} = (-2, -1, 3)$ and $\mathbf{c} = (3, 1, -1)$. Find two different parametric forms (vector equations) for the line which passes through the points $\mathbf{a}$ and $\mathbf{c}$.

Module problems

Problem 2.8 For the following vectors $\mathbf{a}$ and $\mathbf{b}$, find the component of $\mathbf{a}$ along $\mathbf{b}$ and the component of $\mathbf{b}$ along $\mathbf{a}$: (i) $\mathbf{a} = (2, 1)$, $\mathbf{b} = (1, 2)$; (ii) $\mathbf{a} = (-1, 3, 1)$, $\mathbf{b} = (2, -1, -2)$.

Problem 2.9 (a) Determine all vectors in $\mathbb{R}^2$ which are perpendicular to $(3, -1)$.

(b) Find (i) in the form $ax + by = c$ and (ii) in parametric form the equation of the line through the point $(2, 1)$ which is perpendicular to the vector $(3, -1)$.

Problem 2.10 Find, in the form $ax + by + cz = d$ the equation of the plane through the point $(2, -1, 5)$ which is perpendicular to the vector $(2, -3, -4)$. Does this plane pass through the origin?

Problem 2.11 Let L be the line through the origin and the point $(1, 2, -2)$ and let $\mathbf{a}$ be the point $(-1, 2, 5)$.

(a) Find a unit vector $\mathbf{e}$ so that L has parametric form $\mathbf{x} = t\mathbf{e}$.

(b) Find the components of $\mathbf{a}$ (i) along $\mathbf{e}$ and (ii) perpendicular to $\mathbf{e}$.

(c) Find the closest point $\mathbf{p}$ of L to $\mathbf{a}$ (this is the point obtained from $\mathbf{a}$ by projecting onto the line L).

(d) Calculate the distance from $\mathbf{a}$ to the line L.

Problem 2.12 Let L be the line through the points $(-1, 2, 4)$ and $(1, 1, 2)$. Let $\mathbf{a}$ be the point $(3, -1, 4)$.

(a) Find a vector $\mathbf{b}$ and a unit vector $\mathbf{e}$ so that L has parametric form $\mathbf{x} = \mathbf{b} + t\mathbf{e}$.

(b) Find the components of $\mathbf{a} - \mathbf{b}$ (i) along $\mathbf{e}$ and (ii) perpendicular to $\mathbf{e}$.

(c) Find the closest point $\mathbf{p}$ of L to $\mathbf{a}$.

(d) Calculate the distance from $\mathbf{a}$ to the line L.

Problem 2.13 Let $\mathbf{b}$ be the vector $(2, -3, 6)$ and let $\mathbf{a}$ be the point $(4, -2, 1)$. Let M be the plane through the origin perpendicular to $\mathbf{b}$.

(a) Find the unit vector $\mathbf{e}$ in the direction of $\mathbf{b}$.

(b) Find the components of $\mathbf{a}$ (i) along $\mathbf{e}$ and (ii) perpendicular to $\mathbf{e}$.

(c) Find the point $\mathbf{p}$ which is the closest point of M to $\mathbf{a}$.

(d) Calculate the distance from $\mathbf{a}$ to the plane M.

Problem 2.14 Let L be the line passing through the origin and the point $\mathbf{a} = (3, 2, 1)$. Let M be the plane given by the equation $x + 3y - 2z = 0$. Find the angle between the line L and the plane M. (Your answer should be an angle between 0 and $\pi/2$ radians.)

Problem 2.15 Consider the vectors $\mathbf{a} = (-1, 2, -2)$, $\mathbf{b} = (2, 3, -1)$. Find the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Check that your answer is sensible by calculating $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{c}$.

Problem 2.16 Let $\mathbf{a} = (1, 2, -1)$ and $\mathbf{b} = (2, -1, 3)$. Let M be the plane through the origin, $\mathbf{a}$ and $\mathbf{b}$.

(a) Find a normal vector to M.

(b) Find a vector in M perpendicular to $\mathbf{b}$.

Problem 2.17 Consider the vectors $\mathbf{a} = (1, 2, -1)$, $\mathbf{b} = (2, 2, 1)$ and $\mathbf{c} = (-1, -1, 2)$. Calculate the scalar triple products $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ and $\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$ (two of these three should be equal). Do these three vectors lie on some plane through the origin?

Problem 2.18 Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. Suppose that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. Show that, as claimed in the notes, we must have $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (and so also $\mathbf{b} \times \mathbf{a} = \mathbf{0}$). Can this happen without either of $\mathbf{a}$, $\mathbf{b}$ being the zero vector?

Problem 2.19 Using the fact that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$ and other standard properties of the dot product, prove the parallelogram law (stated in the notes):

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Problem 2.20 Let $\mathbf{e}$ be a unit vector in $\mathbb{R}^3$ and let $\mathbf{b}$ be a point of $\mathbb{R}^3$. Let M be the plane through $\mathbf{b}$ and perpendicular to the vector $\mathbf{e}$. Now let $\mathbf{a}$ be any point of $\mathbb{R}^3$, and let $\mathbf{p}$ be the closest point of M to $\mathbf{a}$ (the point obtained from $\mathbf{a}$ by projecting, orthogonally, onto M). Show that $\mathbf{a} - \mathbf{p} = ((\mathbf{a} - \mathbf{b}) \cdot \mathbf{e})\mathbf{e}$ and that the distance from $\mathbf{a}$ to the plane M is $|(\mathbf{a} - \mathbf{b}) \cdot \mathbf{e}|$. [Hint: the rule in the printed notes applies to the case of planes through the origin. You can use this here by first translating everything by subtracting $\mathbf{b}$.]

Problem 2.21 (Distance between two lines) In $\mathbb{R}^3$, most pairs of lines do not meet. In this problem we investigate how we can find the shortest distance between two lines. Suppose that L_1 is the line through the point $\mathbf{a}$ parallel to the vector $\mathbf{v}$ and L_2 is the line through the point $\mathbf{b}$ parallel to the vector $\mathbf{w}$. Suppose further that these lines are not parallel to each other (so $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$.) Now assume that there are points $\mathbf{p}$ on L_1 and $\mathbf{q}$ on L_2 which are as close together as possible. (The existence of these points $\mathbf{p}$ and $\mathbf{q}$ is intuitively obvious.)

(a) Explain why $\mathbf{q} - \mathbf{p}$ must be perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(b) Let $\mathbf{e}$ be a unit vector in the direction of $\mathbf{v} \times \mathbf{w}$. Show that $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{e} = (\mathbf{q} - \mathbf{p}) \cdot \mathbf{e}$.

(c) Show that $\mathbf{q} - \mathbf{p} = ((\mathbf{b} - \mathbf{a}) \cdot \mathbf{e})\mathbf{e}$ (note that this is the component of $\mathbf{b} - \mathbf{a}$ along $\mathbf{v} \times \mathbf{w}$.)

(d) Deduce that the distance between the two lines, $\|\mathbf{q} - \mathbf{p}\|$, is equal to

$$\frac{|(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|}.$$

(d) Notice that we have found this distance in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ and $\mathbf{w}$ without actually finding the points $\mathbf{p}$ and $\mathbf{q}$. However, we did find the vector $\mathbf{c} = \mathbf{q} - \mathbf{p}$. If you translate the line L_1 through the vector $\mathbf{c}$ you get a new line L_3 through $\mathbf{a} + \mathbf{c}$ parallel to $\mathbf{v}$. Do the lines L_2 and L_3 intersect? If so, where?

(e) Find the distance between the line through $(1, 1, 2)$ parallel to $(1, 2, -1)$ and the line through $(-1, 2, 3)$ parallel to $(2, 1, 1)$.

MAPLE exercise

Use MAPLE to check your answers to as many of the above problems as possible.

3 Matrix Algebra

Module problems

Problem 3.1 Let $\mathbf{a} = (3, 2, -1, 2)$ and $\mathbf{b} = (-1, 4, 2, -3)$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b}$.

Problem 3.2 (a) Let $\mathbf{a} = (1 - 2i, -2 + i)$, $\mathbf{b} = (i, 1 + 2i)$ and $z = 1 - i$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $z\mathbf{a}$, $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{a}$.

(b) Let $\mathbf{a} = (1 + i, 2 - i, -1 + 2i, i)$ and $\mathbf{b} = (3 + i, 3 - 2i, 5i, 1 + 2i)$. Find $\mathbf{a} + \mathbf{b}$. Is there a complex scalar z such that $z\mathbf{a} = \mathbf{b}$?

Problem 3.3 Find the matrix which describes the transformation of $\mathbb{R}^2$ given by reflection in the line $y = x$.

Problem 3.4 Consider the matrix

$$A = [a_{ij}] = \begin{bmatrix} -5 & 4 & 1 \\ 2 & -3 & -1 \end{bmatrix}.$$

(a) The matrix A is an $m \times n$ matrix: what are m and n here?

(b) Write down all the rows and columns of A (these are elements of $\mathbb{R}^n$ or $\mathbb{R}^m$).

(c) For this matrix, what are (i) a_{12} (ii) a_{21}?

Problem 3.5 (a) Consider the matrices A, B given by

$$A = \begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ -1 & -5 \end{bmatrix}.$$

Find (i) $A + B$ (ii) $A - B$ (iii) $2A - 3B$.

(b) Find

$$(2 + i)\begin{bmatrix} 1 - i & 1 + i \\ 2 - i & 3i \end{bmatrix} + (1 + i)\begin{bmatrix} -i & 2 \\ 1 + 3i & -1 \end{bmatrix}.$$

Problem 3.6 Calculate the following products.

$$\begin{bmatrix} 1 & -1 & 3 \\ -2 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix};$$

$$\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 2 \\ 1 + i & 2 - 3i \\ i & 1 \end{bmatrix}\begin{bmatrix} -i \\ 1 - i \end{bmatrix}.$$

Problem 3.7 (a) For the following matrices decide which products of two of them exist (including the possibility that a matrix might be multiplied by itself) and calculate these products.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ -1 & -3 \\ 1 & 2 \end{bmatrix}.$$

(b) Calculate the following matrix products:

$$\begin{bmatrix} i & 2 \\ 1-2i & 0 \end{bmatrix}\begin{bmatrix} 2+i & 0 \\ -2 & 3i \end{bmatrix}; \quad [1-i \quad i]\begin{bmatrix} 1+i \\ 2-i \end{bmatrix}; \quad \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}[1-i \quad i].$$

Problem 3.8 Determine which of the following 2×2 matrices are invertible (non-singular) and which are singular. Find the inverses of the ones which are invertible.

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}; \quad \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}; \quad \begin{bmatrix} i & 2 \\ -1 & 3i \end{bmatrix}; \quad \begin{bmatrix} 1+i & 2 \\ i & 1+i \end{bmatrix}.$$

Problem 3.9 Which, if any, of the following matrices A, B, C are (i) symmetric, (ii) Hermitian, (iii) orthogonal, (iv) unitary?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Problem 3.10 Show that the only square matrices which are both triangular and symmetric are the diagonal matrices. Deduce that the square zero matrices are the only square matrices which are symmetric and strictly triangular.

Problem 3.11 For each of the matrices A, B, find (i) the matrix of cofactors (ii) the adjugate matrix (iii) the determinant and (iv) the inverse, if it exists:

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} i & 0 & 1 \\ 0 & 2i & 2 \\ 3i & -i & 0 \end{bmatrix}.$$

Problem 3.12 Using short-cuts where possible, evaluate the following determinants (you may find it helpful to use row and/or column operations!)

(i) $\begin{vmatrix} -5 & -3 \\ -2 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 0 & 1 \\ 5 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 & -1 \\ 3 & 4 & 1 & 2 \\ 5 & 3 & 1 & 2 \\ 3 & 4 & 1 & 5 \end{vmatrix}$ (iv) $\begin{vmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & -2 & -5 \\ 3 & 4 & -1 & 15 \\ -2 & 0 & -1 & -20 \end{vmatrix}$

(v) $\begin{vmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4i \\ -3 & 0 & 0 & 0 & 0 \end{vmatrix}$

Problem 3.13 For which values of t, if any, is the following matrix singular?

$$\begin{bmatrix} 1 & -1 & t-1 \\ 2 & t & -4 \\ 0 & t+2 & -8 \end{bmatrix}$$

Problem 3.14 Consider the transformation of $\mathbb{R}^2$ obtained by multiplying by the matrix $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$.

(a) This transformation maps the unit square to some parallelogram in $\mathbb{R}^2$. What is the area of this parallelogram?

(b) The same transformation maps the circle centred on the origin and with radius 1 to some ellipse. What is the area of this ellipse? [You need not find a formula for the ellipse: instead you should quote a fact mentioned in the notes.]

Problem 3.15 Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector in $\mathbb{R}^3$. Consider the transformation of $\mathbb{R}^3$ given by $T(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$. Find a matrix A such that this transformation has the form $T(\mathbf{x}) = A\mathbf{x}$. (Remember that we think of vectors in $\mathbb{R}^3$ as column vectors.)

Problem 3.16 Let A be an invertible $n \times n$ matrix. Show that there is no non-zero $n \times n$ matrix B such that AB is the $n \times n$ zero matrix.

MAPLE exercise

Use MAPLE to check your answers to as many of the above problems as possible. (WARNING: the MAPLE linalg function dotprod must be used with care for complex vectors. To give the dot product used in the notes, you have to use this in the form `dotprod(a,b,'orthogonal')`. There is no need for this when dealing with real vectors.)

4 Linear Systems

A-level revision

Problem 4.1 (a) Find two real numbers x and y which satisfy simultaneously the equations

$$2x - y = 5$$

and

$$x + 3y = 2.$$

(b) Find all solutions (if any) to the two simultaneous equations

$$2x - 3y + z = 4$$

and

$$3x + y - 2z = 7.$$

Module problems

Problem 4.2 Consider the following system of equations.

$$x - 2y = 4$$

$$2x + y = 3.$$

Set $\mathbf{x} = (x, y)$. Find a 2×2 matrix A and a vector $\mathbf{b}$ so that the system of equations above takes the form

$$A\mathbf{x} = \mathbf{b}.$$

Find A^{-1} and hence find the solution vector $\mathbf{x}$. Check your answer!

Problem 4.3 For the following system of linear equations (three equations in four unknowns) write down the matrix of coefficients A, the constant vector $\mathbf{b}$ and the augmented matrix $[A|\mathbf{b}]$. (You need not solve the system of equations. The matrix A should have 3 rows and 4 columns and the augmented matrix should be 3×5.)

$$x_1 - 3x_2 + 2x_4 = 5,$$

$$-2x_2 + x_3 - 4x_4 = 10,$$

$$-3x_1 - 2x_3 + 7x_4 = -2.$$

Problem 4.4 Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 2 & 3 & 6 \end{array}\right].$$

This augmented matrix corresponds to a system of 2 equations in 4 unknowns. It is already in row echelon form (as defined in the notes).

(a) Choosing appropriate names for the unknowns (variables), write down the system of equations corresponding to this augmented matrix.

(b) Find the leading entries, the pivot columns, the leading variables and the free variables.

(c) Find the solution set for this system of equations.

Problem 4.5 Find the solution sets for the following system of equations.

(a)

$$x_1 + 2x_2 - x_3 + x_4 = 3$$
$$-x_1 + x_2 + 2x_3 + 3x_4 = 4$$
$$x_1 + 5x_2 + 5x_4 = 10.$$

(b)

$$x + 2y - z = 4$$
$$2x + y + z = 6$$
$$5x + 4y + z = 17.$$

Problem 4.6 What are the possible echelon forms for a 2×3 matrix? What reduced echelon forms are possible?

Problem 4.7 For which values of k (if any) does the following system of equations have (i) no solutions (ii) a unique solution (iii) infinitely many solutions?

$$x + 2y + z = 0$$

$$2x + (5+k)y + 6z = 0$$
$$3x + (7+k)y + (11+k)z = 0.$$

Problem 4.8 Use Cramer's rule to solve each of the following systems of linear equations. Check your answers!

(a) (Two equations in two unknowns)

$$3x_1 + x_2 = -1,$$

$$4x_1 - x_2 = -6.$$

(b) (Three equations in three unknowns)

$$x_1 + 2x_2 - x_3 = -1,$$

$$2x_1 - 3x_2 + x_3 = 8,$$

$$x_1 - 2x_2 + 3x_3 = 7.$$

Problem 4.9 Find the inverses of the following matrices A and B (i) using the formula from Chapter 3 (ii) by Gauss-Jordan inversion.

$$A = \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix}.$$

(Which method did you find faster?)

Problem 4.10 Using standard results from the notes, show that for every singular $n \times n$ matrix A there is at least one non-zero $n \times n$ matrix B such that AB is the $n \times n$ zero matrix. [Hint: there is at least one such matrix B where all of the columns are the same.]

MAPLE exercise

Use MAPLE to check your answers to as many of the above problems as possible.

5 Eigenvalues and eigenvectors

Module problems

Problem 5.1 Calculate the eigenvalues and eigenvectors of the following matrices.

$$\text{(a)} \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} \text{(b)} \begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 8 & 3 & -4 \end{bmatrix} \text{(c)} \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix} \text{(e)} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

Problem 5.2 Give an example of a 2×2 matrix A such that A is not the zero matrix and yet the only eigenvalue of A is 0.

PRIZE PROBLEMS

Please note that Dr Feinstein did not invent these prize problems (they are well-known). For each of the prize problems, a prize will be awarded by Dr Feinstein for the first correct solution (including justification sufficient to convince Dr Feinstein!) handed in (by the deadline below) by a student registered for G11LMA: Linear Mathematics in the academic year 2003-4. Solutions should be handed directly to Dr Feinstein on or before May 7 2004. Note that the solutions to these prize problems need not necessarily have anything to do with G11LMA Linear Mathematics.

Prize problem 1. [Prize: 10 pounds] Give (with proof, of course!) an explicit example of a positive real number t such that the sequence $\cos(n!t)$ does not converge.

Prize problem 2. [Prize: 10 pounds] Suppose that a rectangle is formed from finitely many smaller rectangles, and that each of the smaller rectangles has the property that at least one of its sides has integer length. Show that the original rectangle must also have at least one side of integer length.

Prize problem 3. [Prize: 10 pounds] Show that there is no infinite sequence of pairwise disjoint, non-empty closed intervals $[a_n, b_n] \subseteq [0, 1]$ whose union is all of $[0, 1]$.

Prize problem 4. [Prize: 10 pounds] Does there exist an uncountable collection of subsets of $\mathbb{N}$ with the property that, for each pair of sets A, B in the collection, either A is a subset of B or B is a subset of A?

Prize problem 5. [Prize: 15 pounds] Let f be a real-valued function defined on the unit square $[0, 1]^2$. Suppose that, for each fixed $x \in [0, 1]$, $f(x, y)$ is a polynomial in y, and that, for each fixed $y \in [0, 1]$, the function $f(x, y)$ is a polynomial in x. Prove that f must be a polynomial in the two variables x and y.

Prize problem 6. [Prize: 7 pounds] Show that there DOES exist a function f from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$ such that $f(x, y)$ is NOT a polynomial in x and y, and yet for each fixed $x \in \mathbb{Q}$, $f(x, y)$ is a polynomial in y, and, for each fixed $y \in \mathbb{Q}$, the function $f(x, y)$ is a polynomial in x.