G11LMA: Linear Mathematics, Solutions to problems on Autumn Semester material

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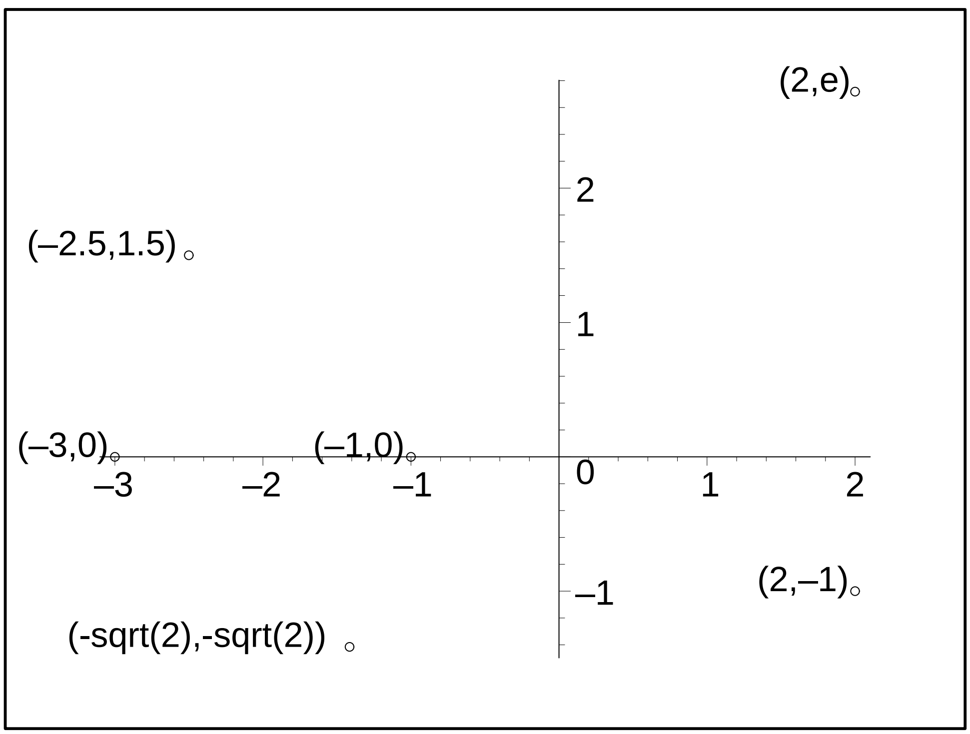
1 Complex Numbers

A-level revision

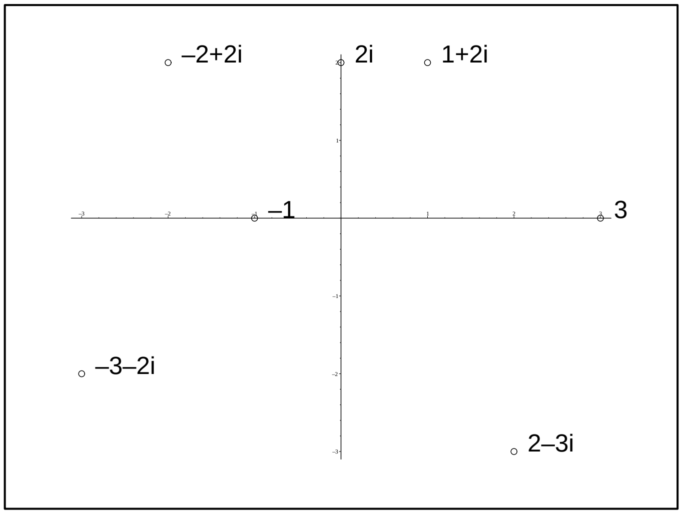
Problem 1.1 Here are the six points in $\mathbb{R}^2$, plotted using MAPLE. Note that `sqrt(2)` is one of the ways to get MAPLE to calculate $\sqrt{2}$.



Six points in $\mathbb{R}^2$, plotted using MAPLE

Module problems

Problem 1.2 Here are the required seven points.



Seven points in $\mathbb{C}$, plotted using MAPLE

Problem 1.3 Let $z = 3 - 2i$ and $w = 1 + 4i$.

(a) We have $z + w = (3 + 1) + (-2 + 4)i = 4 + 2i$. Similarly $z - w = 2 - 6i$. Remembering that $i^2 = -1$ we have

$$z^2 = (3 - 2i)^2 = 3^2 - 2 \times 3 \times (2i) + (2i)^2 = 9 - 12i - 4 = 5 - 12i$$

and

$$zw = (3 \times 1 - (-2) \times 4) + ((-2) \times 1) + 3 \times 4)i = 11 + 10i.$$

(b) Since $z = 3 - 2i$ we have $\bar{z} = 3 + 2i$ (we change the sign of the imaginary part). The formula for $|z|$ gives us

$$|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}.$$

We now verify the formula $z\bar{z} = |z|^2$ in this case:

$$z\bar{z} = (3-2i)(3+2i) = 3^2 - (2i)(3) + 3(2i) - (2i)(2i) = 3^2 - 6i + 6i - (-4) = 13 = |z|^2,$$

as expected.

(c) We can use a standard formula for $1/z$ from the notes (or from above).

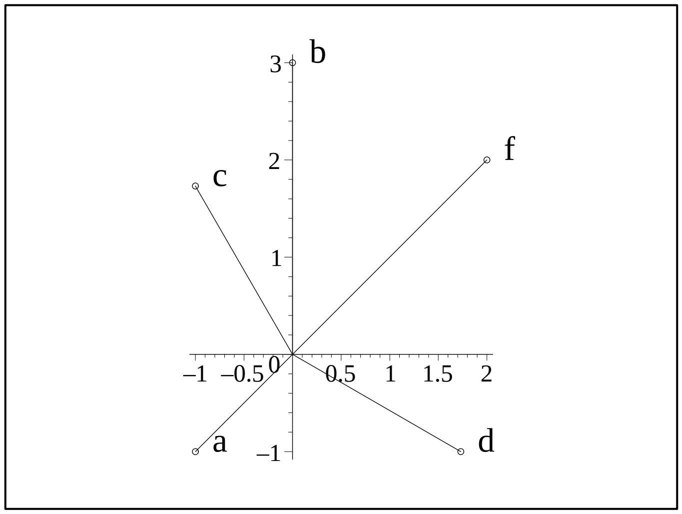
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{3 + 2i}{13} = \frac{3}{13} + \frac{2}{13}i.$$

This gives us $\mathrm{Re}(1/z) = 3/13$ and $\mathrm{Im}(1/z) = 2/13$ (and NOT $(2/13)i$).

Similar calculations give us $1/w = (1/17) - (4/17)i$, so $\mathrm{Re}(1/w) = 1/17$ and $\mathrm{Im}(1/w) = -4/17$. Then $z/w = z(1/w) = (3 - 2i)((1/17) - (4/17)i) = -5/17 - (14/17)i$. So $\mathrm{Re}(z/w) = -5/17$ and $\mathrm{Im}(z/w) = -14/17$. Similarly $w/z = -5/13 + (14/13)i$, so $\mathrm{Re}(w/z) = -5/13$ and $\mathrm{Im}(w/z) = 14/13$.

Problem 1.4 (a) Remember that the principal argument of z, $\mathrm{Arg}(z)$, is an angle measured in radians, with $-\pi < \mathrm{Arg}(z) \leq \pi$. This measures the angle between the positive real axis and the line joining the origin to z. Angles are measured anticlockwise. This means that points z above the real axis will have $0 < \mathrm{Arg}(z) < \pi$ while points below the real axis have $-\pi < \mathrm{Arg}(z) < \pi$. Non-zero points on the real axis have principal argument 0 (for positive real numbers) or π (for negative real numbers).

Here is an Argand diagram showing the five points in question, where $a = -1 - i$, $b = 3i$, $c = -1 + \sqrt{3}i$, $d = \sqrt{3} - i$ and $f = 2 + 2i$.



We now see easily (using standard right-angled triangles) that $\mathrm{Arg}(a) = -\pi/2 - \pi/4 = -3\pi/4$, $\mathrm{Arg}(b) = \pi/2$, $\mathrm{Arg}(c) = \pi/2 + \pi/6 = 2\pi/3$, $\mathrm{Arg}(d) = -\pi/6$ and $\mathrm{Arg}(f) = \pi/4$. To find these numbers in polar and exponential form we need to find their moduli. We obtain $|a| = \sqrt{2}$, $|b| = 3$, $|c| = 2$, $|d| = 2$ and $|e| = \sqrt{8} = 2\sqrt{2}$. So, in polar form,

$$a = \sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4))$$

and in exponential form

$$a = \sqrt{2}\mathrm{e}^{i(-3\pi/4)}.$$

Similarly

$$b = 3(\cos(\pi/2) + i\sin(\pi/2)) = 3\mathrm{e}^{i\pi/2}, \quad c = 2(\cos(2\pi/3) + i\sin(2\pi/3)) = 2\mathrm{e}^{i(2\pi/3)},$$

$$d = 2(\cos(-\pi/6)+i\sin(-\pi/6)) = 2\mathrm{e}^{i(-\pi/6)} \quad \text{and} \quad f = 2\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)) = 2\sqrt{2}\mathrm{e}^{i\pi/4}.$$

(b) Using the notation from part (a) we are asked to find c^7. We have $c = 2\mathrm{e}^{i(2\pi/3)}$ so, by our rule for finding powers of complex numbers,

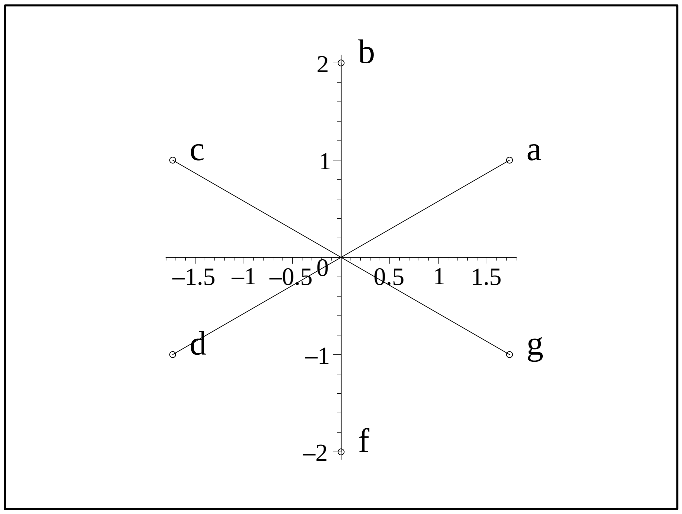
$$c^7 = 2^7\mathrm{e}^{i(7\times 2\pi/3)} = 128\mathrm{e}^{i(14\pi/3)}.$$

Although this answer is correct, if you want to find the principal argument of the answer note that, since $\mathrm{e}^{i(2\pi)} = 1$, we have $\mathrm{e}^{i(14\pi/3)} = \mathrm{e}^{i(2\pi/3)}$ and so the principal argument of c^7 is $2\pi/3$. In the form $x + yi$ we have

$$c^7 = 128(-(1/2) + (\sqrt{3}/2)i)) = -64 + 64\sqrt{3}i.$$

(c) We are asked to find **all** possible arguments for the complex number $d = \sqrt{3} - i$. We know that the principal argument is $-\pi/6$. The other arguments are obtained from the principal argument by adding an integer (possibly negative) multiple of 2π. So the possible arguments for $\sqrt{3} - i$ are $-\pi/6 + 2n\pi$, where $n \in \mathbb{Z}$.

Problem 1.5 First we write -64 in exponential form as $64e^{i\pi}$. We now use the formula given in the notes. In exponential form, the six complex sixth roots of the number $-64 = Re^{i\pi}$ are $64^{1/6}e^{i(2k\pi+\pi)/6}$, $k = 0, 1, \ldots, 5$, i.e. $2e^{i\pi/6}$, $2e^{i\pi/2}$, $2e^{i(5\pi/6)}$, $2e^{i(7\pi/6)}$, $2e^{i(3\pi/2)}$ and $2e^{i(11\pi/6)}$. Note that the last few arguments above are not the **principal** values of the argument. If you wish you can (for example) rewrite $2e^{i(11\pi/6)}$ as $2e^{i(-\pi/6)}$. (This is not required.)
Call these six sixth roots a, b, c, d , f and g respectively. We can plot these points on the Argand diagram and use standard right-angled triangles to find them in the form $x + iy$.



The six complex sixth roots of -64, plotted using MAPLE.

(Note the pattern!) In the form $x + iy$ these six numbers are, respectively, $\sqrt{3} + i$, $2i$, $-\sqrt{3} + i$, $-\sqrt{3} - i$, $-2i$ and $\sqrt{3} - i$.

Problem 1.6 Following the hint, we first note that $2i = 2e^{i\pi/2}$, and the two complex square roots of this number are (using our formula for nth roots with $n = 2$) $\sqrt{2}e^{i\pi/4}$ and $\sqrt{2}e^{i(5\pi/4)}$. However, as always happens with square roots, the second square root is minus the first (remember also that $e^{i\pi} = -1$). So the two square roots are $\pm\sqrt{2}e^{i\pi/4}$. However $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = (1/\sqrt{2}) + (1/\sqrt{2})i$, so the two square roots are simply $1 + i$ and $-1 - i$.
Now we use the quadratic formula to solve the equation

$$z^2 + (3 + 3i)z + 4i = 0 :$$

we have $a = 1$, $b = 3 + 3i$ and $c = 4i$. Informally the two answers are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 - 3i \pm \sqrt{(3 + 3i)^2 - 4i}}{2} = \frac{-3 - 3i \pm \sqrt{2i}}{2}.$$

What this means is that we use the two complex square roots of $2i$ (found above) to give us the two solutions $(-3 - 3i + (1 + i))/2 = -1 - i$ and $(-3 - 3i - (1 + i))/2 = -2 - 2i$.

Problem 1.7 (Note: the function $\arctan$ is also known as $\tan^{-1}$.) If $x < 0$ then $\operatorname{Arg}(x + iy)$ should be either $< -\pi/2$ (if $y < 0$) or $> \pi/2$ (if $y \geq 0$). However, $\arctan$ only takes values strictly between $-\pi/2$ and $\pi/2$, so $\arctan(y/x)$ can not give the correct answer. In fact it will be out by π one way or the other, and so it is not a valid argument at all for $x + iy$.

Problem 1.8 (a) Most pairs of complex numbers will do but we are asked for a specific example. We can take $z = w = (1 + i)$, and then $\operatorname{Re}(zw) = \operatorname{Re}(2i) = 0$ while $\operatorname{Re}(z)\operatorname{Re}(w) = 1 \neq 0$.

(b) From the notes we have

$$\operatorname{Re}(zw) = \operatorname{Re}(z)\operatorname{Re}(w) - \operatorname{Im}(z)\operatorname{Im}(w).$$

It follows that $\operatorname{Re}(zw) = \operatorname{Re}(z)\operatorname{Re}(w)$ if and only if $\operatorname{Im}(z)\operatorname{Im}(w) = 0$. This happens if and only if at least one of $\operatorname{Im}(z)$, $\operatorname{Im}(w)$ is 0 i.e. at least one of the numbers z, w is a real number.

(c) We have $\operatorname{Im}(z) = 1$. Suppose that $w = x + yi$, where x and y are real. Then $\operatorname{Im}(w) = y$, while $zw = (1 + i)(x + yi) = (x - y) + (x + y)i$ and so $\operatorname{Im}(zw) = (x + y)$. Thus $\operatorname{Im}(zw) = \operatorname{Im}(z)\operatorname{Im}(w)$ if and only if $x = 0$, i.e. the number w is purely imaginary (that is, w lies on the imaginary axis).

(d) This time, with $w = x + iy$ as above, $zw = i(x + iy) = -y + ix$. This gives us $\operatorname{Im}(zw) = x$, while $\operatorname{Im}(z)\operatorname{Im}(w) = y$ and so $\operatorname{Im}(zw) = \operatorname{Im}(z)\operatorname{Im}(w)$ if and only if $y = x$. Note that the set of all solutions w forms a straight line through the origin passing through the point $1 + i$ (which was to be expected from our answer to part (c)).

(e) With $w = x + iy$ as above we have $\operatorname{Im}(zw) = ay + bx$ while $\operatorname{Im}(z)\operatorname{Im}(w) = by$. These are equal if and only if $ay + bx = by$ i.e.

$$bx + (a - b)y = 0.$$

If a and $a - b$ are not both 0 this gives a straight line through the origin (which will be vertical if $a = b$ and horizontal if $b = 0$). On the other hand, if $a = (a - b) = 0$ then $a = b = 0$ and $z = 0$. Unsurprisingly, all complex numbers w satisfy the equation in this case!

Problem 1.9 (a) We know, for complex numbers a and b, that $|ab| = |a||b|$. Thus

$$|\exp(z)| = |\mathrm{e}^x \mathrm{e}^{iy}| = |\mathrm{e}^x||\mathrm{e}^{iy}| = \mathrm{e}^x,$$

because e^x is a positive real number and $|\mathrm{e}^{iy}| = 1$.

(b) Write $z = x + iy$ as usual. We have $w = \exp(z)$ so (by (a)) $|w| = \mathrm{e}^x > 0$. Thus $w \neq 0$, and $x = \log(|w|)$, i.e. $\operatorname{Re}(z) = \log(|w|)$, as required. Finally, set $r = |w| = \mathrm{e}^x$. Then $w = \exp(z) = re^{iy}$, and so x$\operatorname{Im}(z) = y$ is an argument of w.

(c) First note that the steps in (b) are reversible: for $w \neq 0$, if $x = \log(|w|)$ and y is an argument of w then $\exp(z) = w$. Now write $w = 1 + i$ in exponential form as $\sqrt{2}e^{i\pi/4}$. By above, the complex logarithms of w are the numbers $z = x + yi$ where $x = \log(|w|) = \log(\sqrt{2})$, and y is an argument for w, so $y = \pi/4 + 2k\pi$ where $k \in \mathbb{Z}$.

(d) As above, the complex logarithms of w have the form $x + iy$ where $x = \log(|w|)$ and y is an argument of w. Since w has infinitely many different arguments, differing from each other by integer multiples of 2π, w has infinitely many different complex logarithms, differing from each other by integer multiples of $2\pi i$.

2 Vector Algebra and Geometry

A-level revision

Problem 2.1 We have $\mathbf{a} = (3, -2)$ and $\mathbf{b} = (1, 2)$. This gives $\mathbf{a} + \mathbf{b} = (4, 0)$, $\mathbf{a} - \mathbf{b} = (2, -4)$, $3\mathbf{a} = (9, -6)$, $2\mathbf{b} = (2, 4)$ and $3\mathbf{a} - 2\mathbf{b} = (7, -10)$.

Problem 2.2 We have $\mathbf{a} = (-1, 2, 2)$ and $\mathbf{b} = (3, -4, 7)$. This gives $\mathbf{a} + \mathbf{b} = (2, -2, 9)$ and $2\mathbf{a} - 3\mathbf{b} = (-11, 16, -17)$.

Problem 2.3 (a) The distance from $(3, -2)$ to $(-1, 3)$ is $\sqrt{(-1-3)^2 + (3-(-2))^2} = \sqrt{41}$.

(b) The distance from $(2, -1, 3)$ to $(-1, 3, 4)$ is $\sqrt{(-1-2)^2 + (3-(-1))^2 + (4-3)^2} = \sqrt{26}$.

(c) (i) $\|(-2, 5)\| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$, (ii) $\|(-3, 2, -1)\| = \sqrt{(-3)^2 + 2^2 + (-1)^2} = \sqrt{14}$.

Problem 2.4 Recall that the unit vector in the direction of a (non-zero) vector $\mathbf{x}$ is $\widehat{\mathbf{x}} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$. We have $\mathbf{a} = (-5, 12)$, $\mathbf{b} = (4, 5)$ (in $\mathbb{R}^2$) and $\mathbf{c} = (1, -2, 4)$ (in $\mathbb{R}^3$). This gives us $\|\mathbf{a}\| = 13$, $\|\mathbf{b}\| = \sqrt{41}$ and $\|\mathbf{c}\| = \sqrt{21}$. The unit vectors $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ in the directions of, respectively, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are thus $(-5/13, 12/13)$, $(4/\sqrt{41}, 5/\sqrt{41})$ and $(1/\sqrt{21}, -2/\sqrt{21}, 4/\sqrt{21})$.

Problem 2.5 (i) With $\mathbf{a} = (2, -1)$, $\mathbf{b} = (2, 2)$ we have $\|\mathbf{a}\| = \sqrt{5}$, $\|\mathbf{b}\| = \sqrt{8}$ and $\mathbf{a} \cdot \mathbf{b} = 2 \times 2 + (-1) \times 2 = 2$. Using the formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ we obtain

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{2}{\sqrt{5}\sqrt{8}} = \frac{1}{\sqrt{10}}.$$

As this is not 0, the vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular.

(ii) This time $\mathbf{a} = (2, 3)$, $\mathbf{b} = (-3, 2)$. We have $\|\mathbf{a}\| = \sqrt{13}$, $\|\mathbf{b}\| = \sqrt{13}$ and $\mathbf{a} \cdot \mathbf{b} = 0$. This gives us $\cos\theta = 0$: these two vectors are perpendicular.

(iii) With $\mathbf{a} = (-1, 3, 2)$ and $\mathbf{b} = (4, 2, -1)$ we have $\|\mathbf{a}\| = \sqrt{14}$, $\|\mathbf{b}\| = \sqrt{21}$, $\mathbf{a} \cdot \mathbf{b} = 0$. This gives us $\cos\theta = 0$ and so these two vectors are perpendicular.

(iv) When $\mathbf{a} = (2, 1, -1)$ and $\mathbf{b} = (3, 1, 2)$ we have $\|\mathbf{a}\| = \sqrt{6}$, $\|\mathbf{b}\| = \sqrt{14}$ and $\mathbf{a} \cdot \mathbf{b} = 5$. This gives us

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{5}{\sqrt{6}\sqrt{14}} = \frac{5}{2\sqrt{21}}.$$

As this is not 0, these two vectors are not perpendicular.

Problem 2.6 The line passes through the points $(3, 4)$ and $(-2, 1)$ in $\mathbb{R}^2$.
We first find the gradient (slope) of the line: $m = (1-4)/(-2-3) = 3/5$. We then find c by checking either point (or both to check the answer!) For example, we need $4 = (3/5)3 + c$, so $c = 11/5$. So the line we want is

$$y = \frac{3}{5}x + \frac{11}{5}.$$

Problem 2.7 There are infinitely many correct answers here: for example we can write the line as $\mathbf{x} = \mathbf{a} + t(\mathbf{c} - \mathbf{a})$ i.e.

$$\mathbf{x} = (-2, -1, 3) + t(5, 2, -4).$$

Or we could use $\mathbf{x} = \mathbf{c} + s(\mathbf{a} - \mathbf{c})$, i.e.

$$\mathbf{x} = (3, 1, -1) + s(-5, -2, 4).$$

Module problems

Problem 2.8 Remember that the component of $\mathbf{a}$ along $\mathbf{b}$ is equal to $(\mathbf{a} \cdot \widehat{\mathbf{b}})\widehat{\mathbf{b}}$ (where $\widehat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$). This is also equal to

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}.$$

Similarly, the component of $\mathbf{b}$ along $\mathbf{a}$ is equal to both $(\mathbf{b} \cdot \widehat{\mathbf{a}})\widehat{\mathbf{a}}$ and $\dfrac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$. This gives us two ways to find each answer, but the second may be easier by hand since you never need to bring in any square roots.

(i) Here $\mathbf{a} = (2, 1)$ and $\mathbf{b} = (1, 2)$. We have $\|\mathbf{a}\|^2 = 5$, $\|\mathbf{b}\|^2 = 5$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 4$. Using the formulae above, the component of $\mathbf{a}$ along $\mathbf{b}$ is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{5}(1, 2) = (4/5, 8/5).$$

Similarly, the component of $\mathbf{b}$ along $\mathbf{a}$ is

$$\frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{4}{5}(2, 1) = (8/5, 4/5).$$

(ii) This time $\mathbf{a} = (-1, 3, 1)$, $\mathbf{b} = (2, -1, -2)$. We have $\|\mathbf{a}\|^2 = 11$, $\|\mathbf{b}\|^2 = 9$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = -7$. Then the component of $\mathbf{a}$ along $\mathbf{b}$ is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{-7}{9}(2, -1, -2) = (-14/9, 7/9, 14/9),$$

while the component of $\mathbf{b}$ along $\mathbf{a}$ is

$$\frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{-7}{11}(-1,3,1) = (7/11, -21/11, -7/11).$$

Problem 2.9 (a) Consider a vector $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$. Then $\mathbf{x}$ is perpendicular to $(3, -1)$ if and only if $(3, 1)\cdot\mathbf{x} = 0$, i.e. $3x - y = 0$. The set of all vectors solving this gives us the line $y = 3x$ in $\mathbb{R}^2$. (This line passes through the origin, and also the vectors $(1, 3)$ and $(-1, -3)$. These last two are two particularly easy choices for vectors perpendicular to $(3, -1)$.

(b) (i) The line is perpendicular to the vector $(3, -1)$ so has the form $(3, -1)\cdot\mathbf{x} = c$ i.e. $3x - y = c$. The line passes through the point $(2, 1)$, so $c = 5$. The line we want, then, is the line $3x - y = 5$.

(ii) To describe the line in parametric form we need a point on the line (e.g. $(2, 1)$) and a vector giving the direction of the line (e.g. $(1, 3)$, perpendicular to $(3, -1)$ by part (a)). One parametric form for the line is

$$\mathbf{x} = (2, 1) + t(1, 3).$$

Problem 2.10 The plane is perpendicular to the vector $(2, -3, -4)$ and so has the form $(2, -3, -4)\cdot\mathbf{x} = d$, i.e. $2x - 3y - 4z = d$. The plane passes through the point $(2, -1, 5)$, so $d = (2, -3, -4)\cdot(2, -1, 5) = -13$. The equation of the plane is thus

$$2x - 3y - 4z = -13.$$

Since $d = -13 \neq 0$, this plane does not pass through the origin.

Problem 2.11 Let L be the line through the origin and the point $(1, 2, -2)$ and let $\mathbf{a}$ be the point $(-1, 2, 5)$.

(a) We take $\mathbf{e}$ to be the unit vector in the direction of $(1, 2, -2)$. Since $\|(1, 2, -2)\| = 3$, we have $\mathbf{e} = (1/3, 2/3, -2/3)$. (Alternatively you can use minus this vector.)

(b) (i) The component of $\mathbf{a}$ along $\mathbf{e}$ is simply $(\mathbf{a}\cdot\mathbf{e})\mathbf{e}$ (since $\mathbf{e}$ is a unit vector). We have $\mathbf{a}\cdot\mathbf{e} = -7/3$, so the component of $\mathbf{a}$ along $\mathbf{e}$ is $(-7/9, -14/9, 14/9)$. As in an earlier question, this is also equal to

$$\frac{\mathbf{a}\cdot(1, 2, -2)}{\|(1, 2, -2)\|^2}(1, 2, -2).$$

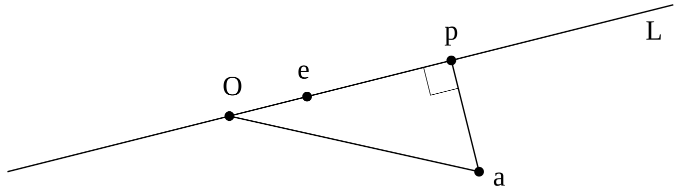
(ii) The component of $\mathbf{a}$ perpendicular to $\mathbf{e}$ is obtained by subtracting from $\mathbf{a}$ the component of $\mathbf{a}$ along $\mathbf{e}$, giving us the answer $(-1, 2, 5) - (-7/9, -14/9, 14/9) = (-2/9, 32/9, 31/9)$.

(c) Since the line passes through the origin, $\mathbf{p}$ is equal to the component of $\mathbf{a}$ along $\mathbf{e}$, i.e $\mathbf{p} = (-7/9, -14/9, 14/9)$.

(d) The distance from $\mathbf{a}$ to the line L is

$$\|\mathbf{a}-\mathbf{p}\| = \|(-2/9, 32/9, 31/9)\| = \frac{\sqrt{2^2+32^2+31^2}}{9} = \frac{\sqrt{221}}{3}.$$

The following figure (not to scale) illustrates this situation.



Problem 2.12 (a) As usual there are infinitely many correct choices for $\mathbf{b}$ (a point on the line) and two possible choices for $\mathbf{e}$. We may take $\mathbf{b} = (-1, 2, 4)$ and $\mathbf{e}$ to be a unit vector in the direction of the vector $(1, 1, 2) - (-1, 2, 4) = (2, -1, -2)$: this gives us $\mathbf{e} = (2/3, -1/3, -2/3)$.

(b) We have $\mathbf{a} - \mathbf{b} = (4, -3, 0)$. The component of $\mathbf{a} - \mathbf{b}$ along $\mathbf{e}$ is

$$((\mathbf{a}-\mathbf{b})\cdot\mathbf{e})\mathbf{e} = (11/3)\mathbf{e} = (22/9, -11/9, -22/9).$$

The component of $\mathbf{a} - \mathbf{b}$ perpendicular to $\mathbf{e}$ is $(4, -3, 0) - (22/9, -11/9, -22/9) = (14/9, -16/9, 22/9)$.

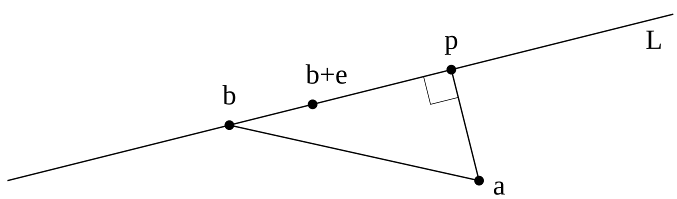
(c) This time the point $\mathbf{p}$ is obtained by adding the component of $\mathbf{a} - \mathbf{b}$ along $\mathbf{e}$ to $\mathbf{b}$. This gives us

$$\mathbf{p} = (-1, 2, 4) + (22/9, -11/9, -22/9) = (13/9, 7/9, 14/9).$$

(d) The distance from $\mathbf{a}$ to the line L is

$$\|\mathbf{a}-\mathbf{p}\| = \|(14/9, -16/9, 22/9)\| = (2/9)\sqrt{7^2+(-8)^2+11^2} = (2/3)\sqrt{26}.$$

A very similar diagram applies (again not to scale). This time the line passes through $\mathbf{b}$ and not through the origin.



Problem 2.13 Let $\mathbf{b}$ be the vector $(2, -3, 6)$ and let $\mathbf{a}$ be the point $(4, -2, 1)$. Let M be the plane through the origin perpendicular to $\mathbf{b}$.

(a) Here $\|\mathbf{b}\| = \sqrt{49} = 7$, so $\mathbf{e} = (2/7, -3/7, 6/7)$.

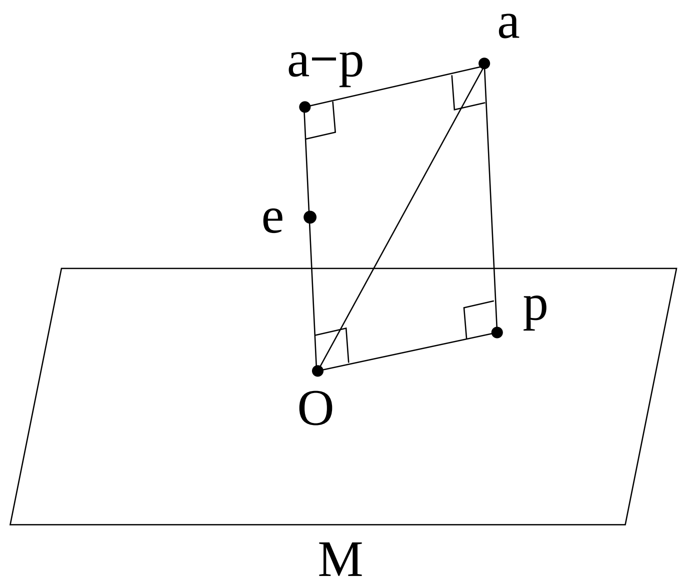
(b) We have $\mathbf{a} \cdot \mathbf{e} = 20/7$, so the component of $\mathbf{a}$ along $\mathbf{e}$ is
$(20/7)\mathbf{e} = (40/49, -60/49, 120/49)$. The component of $\mathbf{a}$ perpendicular to $\mathbf{e}$ is
$(4, -2, 1) - (40/49, -60/49, 120/49) = (156/49, -38/49, -71/49)$.

(c) This time the closest point p is the component of $\mathbf{a}$ perpendicular to $\mathbf{e}$, i.e.
$\mathbf{p} = (156/49, -38/49, -71/49)$.

(d) The distance from $\mathbf{a}$ to the plane M is

$$\|\mathbf{a} - \mathbf{p}\| = \|(20/7)\mathbf{e}\| = 20/7.$$

The following diagram illustrates the situation.



Problem 2.14 The quickest route to the answer is to find the angle between L and a normal vector to M. There are various possibilities, but only one lying between 0 and $\pi/2$. Subtract this angle from $\pi/2$.
The direction of the line L is given by the vector $\mathbf{a} = (3, 2, 1)$.
The equation of the plane M may be rewritten as $(1, 3, -2) \cdot \mathbf{x} = 0$: the vector $\mathbf{b} = (1, 3, -2)$ is normal to this plane.
We now look at the angle between $\mathbf{a}$ and $\pm\mathbf{b}$. At least one choice of signs will give us an angle θ between 0 and $\pi/2$. We detect this by ensuring that $\cos\theta \geq 0$. Since $\mathbf{a} \cdot \mathbf{b} = 7 > 0$, we do not need the minus sign, and we have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{7}{\sqrt{14}\sqrt{14}} = \frac{1}{2}.$$

From this and the standard fact that $\cos(\pi/3) = 1/2$ we deduce that $\theta = \pi/3$. We now subtract θ from $\pi/2$ to obtain $\pi/6$: this is the angle between L and M. (The diagram from the previous question applies here, with $\mathbf{e} = \widehat{\mathbf{b}}$.)

Problem 2.15 Applying the formula,

$$\mathbf{a} \times \mathbf{b} = (2 \times (-1) - (-2) \times 3, (-2) \times 2 - (-1) \times (-1), (-1) \times 3 - 2 \times 2) = (4, -5, -7),$$

i.e $\mathbf{c} = (4, -5, -7)$ We now check that $\mathbf{c}$ really is perpendicular to $\mathbf{a}$ and to $\mathbf{b}$:
$(-1, 2, -2) \cdot (4, -5, -7) = -4 - 10 + 14 = 0$ and $(2, 3, -1) \cdot (4, -5, -7) = 8 - 15 + 7 = 0$, as expected.

Problem 2.16 (a) The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and so is normal to the plane M. Here

$$(1, 2, -1) \times (2, -1, 3) = (5, -5, -5)$$

so we can use the vector $(5, -5, -5)$ or any non-zero multiple of this: for example we can take $\mathbf{n} = (1, -1, -1)$ as our normal vector.

(b) The plane M can now be defined by the equation

$$\mathbf{n} \cdot \mathbf{x} = 0.$$

We want to find a vector in M perpendicular to $\mathbf{b}$. This vector must also be perpendicular to $\mathbf{n}$, so the obvious choice is to take $\mathbf{n} \times \mathbf{b}$ (we can also use any non-zero multiple of this vector). This gives us one possible answer

$$(1, -1, -1) \times (2, -1, 3) = (-4, -5, 1).$$

Problem 2.17 We have $\mathbf{a} = (1, 2, -1)$, $\mathbf{b} = (2, 2, 1)$ and $\mathbf{c} = (-1, -1, 2)$. This gives us $\mathbf{b} \times \mathbf{c} = (5, -5, 0)$, $\mathbf{c} \times \mathbf{a} = (-3, 1, -1)$ and $\mathbf{b} \times \mathbf{a} = (-4, 3, 2)$. From this we obtain

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (1, 2, -1) \cdot (5, -5, 0) = -5,$$

$$\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = (2, 2, 1) \cdot (-3, 1, -1) = -5$$

and

$$\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = (-1, -1, 2) \cdot (-4, 3, 2) = 5.$$

Each of these triple products gives plus or minus the volume of the parallelepiped formed by drawing edges from 0 to each of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and then adding edges parallel to these. (Which sign you get depends on whether or not the three vectors form a 'right-handed triple' in the order they occur in the scalar triple product.) Since this volume is 5, and not 0, the three vectors do not lie on any plane through the origin.

Problem 2.18 We are given that $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$ with $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. We also have the standard fact from the notes that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. Adding these equations gives us $2(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ and so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. (It then follows that we also have $\mathbf{b} \times \mathbf{a} = \mathbf{0}$.)
This does not necessarily mean that one or both of $\mathbf{a}$, $\mathbf{b}$ is $\mathbf{0}$. In fact it happens whenever $\mathbf{a}$ and $\mathbf{b}$ are parallel. For a specific example (always recommended!) we can take $\mathbf{a} = \mathbf{b} = (1, 0, 0)$.

Problem 2.19 We have

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot(\mathbf{x}+\mathbf{y})+\mathbf{y}\cdot(\mathbf{x}+\mathbf{y})$$

$$= \mathbf{x}\cdot\mathbf{x}+\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{y}.$$

Since $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x}$, this gives us

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2.$$

Similarly,

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2.$$

Adding these two equations gives the result:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Problem 2.20 We translate the problem to the one dealt with in the notes by subtracting $\mathbf{b}$ from everything. Let M' be the plane through the origin and perpendicular to $\mathbf{e}$ (obtained by subtracting $\mathbf{b}$ from all the points of M) set $\mathbf{a}'=\mathbf{a}-\mathbf{b}$ and let $\mathbf{p}'$ be the point of M' closest to $\mathbf{a}'$. Since translation has no effect on distances, angles or directions, it is clear that we must have $\mathbf{p}'=\mathbf{p}-\mathbf{b}$ and $\mathbf{a}-\mathbf{p}=\mathbf{a}'-\mathbf{p}'$.
The rule in the notes applies to the plane M' and tells us that

$$\mathbf{a}'-\mathbf{p}' = (\mathbf{a}'\cdot\mathbf{e})\mathbf{e}$$

i.e. $\mathbf{a}-\mathbf{p}=((\mathbf{a}-\mathbf{b})\cdot\mathbf{e})\mathbf{e}$. The distance from $\mathbf{a}$ to M is thus

$$\|((\mathbf{a}-\mathbf{b})\cdot\mathbf{e})\mathbf{e}\| = |(\mathbf{a}-\mathbf{b})\cdot\mathbf{e}|,$$

as required.
I was also asked about an **alternative proof**, which is fine, based on the following useful fact (which you can quote if you wish): given a vector $\mathbf{c}$ and a non-zero vector $\mathbf{d}$ there is **only one way** to write $\mathbf{c}=\mathbf{c}_1+\mathbf{c}_2$ with $\mathbf{c}_1$ parallel to $\mathbf{d}$ and $\mathbf{c}_2$ perpendicular to $\mathbf{d}$, and this is when $\mathbf{c}_1$ is the component of $\mathbf{c}$ along $\mathbf{d}$ and $\mathbf{c}_2$ is the component of $\mathbf{c}$ perpendicular to $\mathbf{d}$.
Can you prove this? (To say that $\mathbf{c}_1$ is parallel to $\mathbf{d}$ means that there is some $t\in\mathbb{R}$ with $\mathbf{c}_1=t\mathbf{d}$.)
In this question we know that $\mathbf{a}-\mathbf{p}$ is perpendicular to M, and so it must be parallel to $\mathbf{e}$. We have $\mathbf{a}-\mathbf{b}=(\mathbf{a}-\mathbf{p})+(\mathbf{p}-\mathbf{b})$. The first is parallel to $\mathbf{e}$ and the second is perpendicular to $\mathbf{e}$, so (by the fact above) the first term on the right hand side must be the component of $\mathbf{a}-\mathbf{b}$ along $\mathbf{e}$ and the second is the component of $\mathbf{a}-\mathbf{b}$ perpendicular to $\mathbf{e}$. Thus $\mathbf{a}-\mathbf{p}=((\mathbf{a}-\mathbf{b})\cdot\mathbf{e})\mathbf{e}$ as in the first proof above, and the rest follows.

Problem 2.21 (Distance between two lines) We are given that L_1 is the line through the point $\mathbf{a}$ parallel to the vector $\mathbf{v}$ and L_2 is the line through the point $\mathbf{b}$ parallel to the vector $\mathbf{w}$. Suppose further that these lines are not parallel to each other (so $\mathbf{v}\times\mathbf{w}\neq\mathbf{0}$.) Now assume that there are points $\mathbf{p}$ on L_1 and $\mathbf{q}$ on L_2 which are as close together as possible. (The existence of these points $\mathbf{p}$ and $\mathbf{q}$ is intuitively obvious.)

(a) Since these points are as close as possible together, we have (in particular) both that $\mathbf{q}$ is the closest point of L_2 to $\mathbf{p}$ and $\mathbf{p}$ is the closest point of L_1 to $\mathbf{q}$. The first of these tells us that $\mathbf{p}-\mathbf{q}$ is perpendicular to L_2 and the second tells us that $\mathbf{q}-\mathbf{p}$ is perpendicular to L_1. In other words $\mathbf{q}-\mathbf{p}$ must be perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, as claimed.

(b) Let $\mathbf{e}$ be a unit vector in the direction of $\mathbf{v}\times\mathbf{w}$. Show that $(\mathbf{b}-\mathbf{a})\cdot\mathbf{e}=(\mathbf{q}-\mathbf{p})\cdot\mathbf{e}$.

Using the vector equations of the lines L_1 and L_2 we see that there are real numbers s and t with $\mathbf{b}=\mathbf{q}+t\mathbf{w}$ and $\mathbf{a}=\mathbf{p}+s\mathbf{v}$. Then

$$\mathbf{b}-\mathbf{a}=(\mathbf{q}-\mathbf{p})+t\mathbf{w}-s\mathbf{v}.$$

Since $\mathbf{e}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, taking the dot product with $\mathbf{e}$ eliminates the last two terms to give us

$$(\mathbf{b}-\mathbf{a})\cdot\mathbf{e}=(\mathbf{q}-\mathbf{p})\cdot\mathbf{e}.$$

Alternatively, we may argue geometrically that any components of these vectors along the directions of either line do not contribute to the component along $\mathbf{e}$ (which is perpendicular to both lines).

(c) We know that $\mathbf{q}-\mathbf{p}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$, so $\mathbf{q}-\mathbf{p}$ must be some scalar multiple of $\mathbf{e}$. Since $\mathbf{e}$ is a unit vector, we see that $\mathbf{q}-\mathbf{p}=((\mathbf{q}-\mathbf{p})\cdot\mathbf{e})\mathbf{e}$. Using part (b) we then have

$$\mathbf{q}-\mathbf{p}=((\mathbf{b}-\mathbf{a})\cdot\mathbf{e})\mathbf{e},$$

as required (and this is just the component of $\mathbf{b}-\mathbf{a}$ along $\mathbf{v}\times\mathbf{w}$.)

(d) By (c), the distance between the two lines, $\|\mathbf{q}-\mathbf{p}\|$, is $\|((\mathbf{b}-\mathbf{a})\cdot\mathbf{e})\mathbf{e}\|=|(\mathbf{b}-\mathbf{a})\cdot\mathbf{e}|$. However, $\mathbf{e}=\frac{\mathbf{v}\times\mathbf{w}}{\|\mathbf{v}\times\mathbf{w}\|}$ and so the distance is

$$\frac{|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|},$$

as required.

(d) Translating the line L_1 through the vector $\mathbf{c}$ means adding $\mathbf{c}$ to all the points on the line L_1. In particular, since the point $\mathbf{p}$ is on L_1, the point $\mathbf{p}+\mathbf{c}$ is on L_3. Here $\mathbf{p}+\mathbf{c}=\mathbf{q}$, so the lines L_2 and L_3 intersect at $\mathbf{q}$. This enables us to find the points $\mathbf{q}$ and then $\mathbf{p}$ if we wish.

(e) We set $\mathbf{a}=(1,1,2)$, $\mathbf{v}=(1,2,-1)$, $\mathbf{b}=(-1,2,3)$ and $\mathbf{w}=(2,1,1)$. Our formula gives us that the distance between the two lines is

$$\frac{|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}.$$

We have $\mathbf{b}-\mathbf{a}=(-2,1,1)$ and $\mathbf{v}\times\mathbf{w}=(3,-3,-3)$, so $\|\mathbf{v}\times\mathbf{w}\|=3\sqrt{3}$ and $|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{v}\times\mathbf{w})|=12$. The distance between the lines is thus $12/(3\sqrt{3})=4/\sqrt{3}$.

3 Matrix Algebra

Module problems

Problem 3.1 With $\mathbf{a} = (3, 2, -1, 2)$ and $\mathbf{b} = (-1, 4, 2, -3)$ we have $\mathbf{a} + \mathbf{b} = (2, 6, 1, -1)$, $\mathbf{a} - \mathbf{b} = (4, -2, -3, 5)$, $2\mathbf{a} = (6, 4, -2, 4)$ and $\mathbf{a} \cdot \mathbf{b} = -3 + 8 - 2 - 6 = -3$.

Problem 3.2 (a) With $\mathbf{a} = (1 - 2i, -2 + i)$, $\mathbf{b} = (i, 1 + 2i)$ and $z = 1 - i$, we have
$\mathbf{a} + \mathbf{b} = (1 - i, -1 + 3i)$, $\mathbf{a} - \mathbf{b} = (1 - 3i, -3 - i)$,
$z\mathbf{a} = ((1 - i)(1 - 2i), (1 - i)(-2 + i)) = (-1 - 3i, -1 + 3i)$,
$\mathbf{a} \cdot \mathbf{b} = (1 - 2i)i + (-2 + i)(1 + 2i) = -2 - 2i$ and $\mathbf{a} \cdot \mathbf{a} = (1 - 2i)^2 + (-2 + i)^2 = -8i$.

(b) With $\mathbf{a} = (1 + i, 2 - i, -1 + 2i, i)$ and $\mathbf{b} = (3 + i, 3 - 2i, 5i, 1 + 2i)$, we have $\mathbf{a} + \mathbf{b} = (4 + 2i, 5 - 3i, -1 + 7i, 1 + 3i)$. If there is a complex scalar z such that $z\mathbf{a} = \mathbf{b}$ then, looking at the last coordinate, we must have $zi = 1 + 2i$ and so $z = (1/i)(1 + 2i) = (-i)(1 + 2i) = 2 - i$. However, $\mathbf{b} \neq (2 - i)\mathbf{a}$, (which of the coordinates would be wrong?) and so there is no such scalar z.

Problem 3.3 Reflection in the line $y = x$ takes the point (x, y) to the point (y, x). Writing $(y, x) = (ax + by, cx + dy)$ gives us $a = 0$, $b = 1$, $c = 1$ and $d = 0$ so the matrix we want is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Problem 3.4 We are given the matrix

$$A = [a_{ij}] = \begin{bmatrix} -5 & 4 & 1 \\ 2 & -3 & -1 \end{bmatrix}.$$

(a) The matrix A has 2 rows and 3 columns so it is a 2×3 matrix (the rows come first). This means that $m = 2$ and $n = 3$.

(b) The rows of A are $R_1 = (-5, 4, 1)$ and $R_2 = (2, -3, -1)$ (regarded as elements of $\mathbb{R}^3$). The columns of A are $C_1 = (-5, 2)$, $C_2 = (4, -3)$ and $C_3 = (1, -1)$ (elements of $\mathbb{R}^2$).

(c) (i) a_{12} is the entry in row 1 column 2 (the rows come first) and so $a_{12} = 4$.

(ii) $a_{21} = 2$.

Problem 3.5 (a) We are given

$$A = \begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ -1 & -5 \end{bmatrix}.$$

Our operations of addition, etc., are done entry by entry using corresponding entries.

(i) $A + B = \begin{bmatrix} (-1) + 1 & 4 + (-3) \\ (-3) + (-1) & 2 + (-5) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}$. Similarly,

(ii) $A - B = \begin{bmatrix} -2 & 7 \\ -2 & 7 \end{bmatrix}$ and (iii) $2A - 3B = \begin{bmatrix} -5 & 17 \\ -3 & 19 \end{bmatrix}$.

(b) We have to calculate

$$(2+i)\begin{bmatrix} 1-i & 1+i \\ 2-i & 3i \end{bmatrix} + (1+i)\begin{bmatrix} -i & 2 \\ 1+3i & -1 \end{bmatrix}$$

which is

$$\begin{bmatrix} (2+i)(1-i)+(1+i)(-i) & (2+i)(1+i)+(1+i)(2) \\ (2+i)(2-i)+(1+i)(1+3i) & (2+i)(3i)+(1+i)(-1) \end{bmatrix}.$$

If you do the calculations you should get the answer $\begin{bmatrix} 4-2i & 3+5i \\ 3+4i & -4+5i \end{bmatrix}$.

Problem 3.6

$$\begin{bmatrix} 1 & -1 & 3 \\ -2 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-1)+(-1)(3)+3(2) \\ (-2)(-1)+(-1)(3)+3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix};$$

Similarly,

$$\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 1+i & 2-3i \\ i & 1 \end{bmatrix}\begin{bmatrix} -i \\ 1-i \end{bmatrix} = \begin{bmatrix} 1(-i)+2(1-i) \\ (1+i)(-i)+(2-3i)(1-i) \\ i(-i)+1(1-i) \end{bmatrix} = \begin{bmatrix} 2-3i \\ -6i \\ 2-i \end{bmatrix}.$$

Problem 3.7 (a) We are given

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ -1 & -3 \\ 1 & 2 \end{bmatrix}.$$

So A is a 3×3 matrix, B is a 2×3 matrix and C is a 3×2 matrix. We can multiply two matrices if the first has the same number of columns as the second has rows: if the first matrix is $m \times n$, say, then the second should be of the form $n \times p$ (the product is then an $m \times p$ matrix). We now see that we can form the products AA (also written as A^2), BA, AC, BC, CB and no others. For example,

$$BC = \begin{bmatrix} 3(-2)+1(-1)+0 & 3(1)+1(-3)+0 \\ 0+(-1)(-1)+2(1) & 0+(-1)(-3)+2(2) \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 3 & 7 \end{bmatrix}.$$

Similarly,

$$A^2 = AA = \begin{bmatrix} 1 & -9 & 3 \\ -1 & 4 & -4 \\ 3 & -3 & 4 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & -7 & -1 \\ 2 & -2 & 5 \end{bmatrix},$$

$$AC = \begin{bmatrix} 1 & 10 \\ -3 & -8 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad CB = \begin{bmatrix} -6 & -3 & 2 \\ -3 & 2 & -6 \\ 3 & -1 & 4 \end{bmatrix}.$$

(b) We have

$$\begin{bmatrix} i & 2 \\ 1-2i & 0 \end{bmatrix}\begin{bmatrix} 2+i & 0 \\ -2 & 3i \end{bmatrix} = \begin{bmatrix} i(2+i)+2(-2) & 0+2(3i) \\ 1-2i)(2+i)+0 & 0+0 \end{bmatrix} = \begin{bmatrix} -5+2i & 6i \\ 4-3i & 0 \end{bmatrix};$$

$[\,1-i \quad i\,]\begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$ is the 1×1 matrix

$$[(1-i)(1+i)+i(2-i)] = [3+2i].$$

Finally, $\begin{bmatrix} 1+i \\ 2-i \end{bmatrix}[\,1-i \quad i\,]$ is the 2×2 matrix

$$\begin{bmatrix} (1+i)(1-i) & (1+i)i \\ (2-i)(1-i) & (2-i)i \end{bmatrix} = \begin{bmatrix} 2 & -1+i \\ 1-3i & 1+2i \end{bmatrix}.$$

Problem 3.8 A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible (non-singular) if and only if its determinant, $ad - bc$, is non-zero. If $ad - bc = 0$ then the matrix is singular. If it is invertible, our formula for the inverse of this 2×2 matrix is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For the first matrix, $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, the determinant is $2(5) - 3(1) = 7 \neq 0$. The matrix is invertible, and the inverse is

$$\frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5/7 & -3/7 \\ -1/7 & 2/7 \end{bmatrix}.$$

The second matrix, $\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$, has determinant $-6 - (-6) = 0$, so this matrix is singular and has no inverse.

The third matrix, $\begin{bmatrix} i & 2 \\ -1 & 3i \end{bmatrix}$, has determinant $i(3i) - 2(-1) = -1 \neq 0$. This matrix is invertible, and the inverse is

$$\frac{1}{-1}\begin{bmatrix} 3i & -2 \\ 1 & i \end{bmatrix} = \begin{bmatrix} -3i & 2 \\ -1 & -i \end{bmatrix}.$$

The fourth matrix, $\begin{bmatrix} 1+i & 2 \\ i & 1+i \end{bmatrix}$, has determinant $(1+i)(1+i) - 2i = 0$. This matrix is singular and has no inverse.

Problem 3.9 (i) A matrix is symmetric if it is equal to its own transpose. We have $A^T = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ and $C^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Here the symmetric matrices are A and B.

(ii) A matrix E is Hermitian if it is equal to its own Hermitian adjoint E^* (recall that $E^* = \left(\overline{E}\right)^T = \overline{E^T}$). Since the entries of A are real, $A^* = A^T = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$, but $B^* = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$ and $C^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. The Hermitian matrices here are A and C.

(iii) A matrix E is orthogonal if it is invertible and $E^{-1} = E^T$. We can simply check whether or not $EE^T = I$. We find that $AA^T = \begin{bmatrix} 5 & -4 \\ -4 & 13 \end{bmatrix}$, $BB^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $CC^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. None of these are equal to I, so none of the three matrices are orthogonal.

(iv) A matrix E is unitary if it is invertible and $E^{-1} = E^*$. Again we can simply check whether or not $EE^* = I$. We find that $AA^* = \begin{bmatrix} 5 & -4 \\ -4 & 13 \end{bmatrix}$, $BB^* = \begin{bmatrix} 2 & -2i \\ 2i & 2 \end{bmatrix}$ and $CC^* = I$. The matrix C is the only one of these three which is unitary.

Problem 3.10 If a matrix is lower triangular then all the entries strictly above the leading diagonal are 0. If such a matrix is also symmetric then all the entries strictly below the leading diagonal must also be 0 and so the matrix must be a diagonal matrix. The same argument applies to upper triangular matrices which are symmetric.
For strictly triangular matrices the entries on the diagonal must be 0, so if such a matrix is diagonal it must be a square zero matrix.

Problem 3.11 We have

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 0 & 1 \\ 0 & 2i & 2 \\ 3i & -i & 0 \end{bmatrix}.$$

(i) The matrix of cofactors has ij entry $c_{ij} = (-1)^{i+j}\det(M_{ij})$ where M_{ij} is the ij minor matrix (obtained by deleting the ith row and the jth column of your matrix). For example, for A, the 23 minor matrix M_{23} is obtained by deleting the second row and the third column to give $M_{23} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. This gives

$$c_{23} = (-1)^{2+3}\det(M_{23}) = -\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = -10.$$

The rest of the cofactors of A are found similarly: the matrix of cofactors for A is $\begin{bmatrix} 0 & 5 & 5 \\ -1 & -7 & -10 \\ -1 & 3 & 5 \end{bmatrix}$.

The matrix of cofactors for B is found similarly and is $\begin{bmatrix} 2i & 6i & 6 \\ -i & -3i & -1 \\ -2i & -2i & -2 \end{bmatrix}$.

The adjugate matrix (Adj) is the transpose of the matrix of cofactors. We have

$$\mathrm{Adj}(A) = \begin{bmatrix} 0 & -1 & -1 \\ 5 & -7 & 3 \\ 5 & -10 & 5 \end{bmatrix} \quad \text{and} \quad \mathrm{Adj}(B) = \begin{bmatrix} 2i & -i & -2i \\ 6i & -3i & -2i \\ 6 & -1 & -2 \end{bmatrix}.$$

(iii) There are many ways to find the determinants. For example, since $\mathrm{Adj}(A)A = A\mathrm{Adj}(A) = \det(A)I$ we can calculate just one of the leading diagonal entries of either $\mathrm{Adj}(A)A$ or $A\mathrm{Adj}(A)$: this corresponds to expanding the determinant of A using one of the columns or one of the rows of A. Expanding $\det(A)$ using the first row of A corresponds to finding the top left hand entry of $A\mathrm{Adj}(A)$, and gives us $\det(A) = 1(0) + (-3)(5) + (2)(5) = -5$. Similarly, $\det(B) = i(2i) + 0 + 1(6) = 4$.

(iv) Since both matrices have non-zero determinants they are invertible (non-singular): their inverses are

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A) = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & 7/5 & -3/5 \\ -1 & 2 & -1 \end{bmatrix}$$

and

$$B^{-1} = \frac{1}{\det(B)}\mathrm{Adj}(B) = \begin{bmatrix} i/2 & -i/4 & -i/2 \\ 3i/2 & -3i/4 & -i/2 \\ 3/2 & -1/4 & -1/2 \end{bmatrix}.$$

For larger matrices the method used in this question to find inverses is much too slow, and other methods are needed.

Problem 3.12 (i) $\begin{vmatrix} -5 & -3 \\ -2 & 4 \end{vmatrix} = (-5)(4) - (-3)(-2) = 26.$

(ii) We can expand by the first row:

$$\begin{vmatrix} 3 & 0 & 1 \\ 5 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} = 3(-1) + 9 = 6.$$

Alternatively you could start by subtracting 3 times the third column from the first column, which does not change the determinant: this gives us

$$\begin{vmatrix} 3 & 0 & 1 \\ 5 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 6.$$

(iii) Remember: adding multiples of one row to another or one column to another does not change the determinant (**but** if you swap two rows, or swap two columns, then the sign of the determinant changes). The following calculations follow by first subtracting row 2 from row 4, expanding by row 4, subtracting 2 times column 2 from column 3 and then expanding by row 1. **Note: You should always indicate which row operations you are carrying out: if you make mistakes you will still obtain some marks for method as long as you explain what you are doing**.

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 3 & 4 & 1 & 2 \\ 5 & 3 & 1 & 2 \\ 3 & 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 3 & 4 & 1 & 2 \\ 5 & 3 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 1 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & 1 & 0 \\ 3 & 4 & -7 \\ 5 & 3 & -5 \end{vmatrix} = 3(-1)\begin{vmatrix} 3 & -7 \\ 5 & -5 \end{vmatrix} = 3(-1)(20) = -60.$$

(iv) This time we repeatedly use row operations of the type that do not change the determinant. We first subtract multiples of row 1 from the other rows to introduce 0s in column 1. We then expand by column 1, and repeat this process, eventually arriving at the answer -4.

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & -2 & -5 \\ 3 & 4 & -1 & 15 \\ -2 & 0 & -1 & -20 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 4 & -3 & -14 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & -2 \\ -2 & 2 & 6 \\ 4 & -3 & -14 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 0 & -1 & 4 \\ 0 & 3 & -10 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 4 \\ 3 & -10 \end{vmatrix} = 2(-2) = -4.$$

(v) Remember that swapping two rows changes the sign of the determinant. So we first swap row 1 with row 5, and then swap rows 4 and 5. This results in a matrix whose only non-zero rows are on the diagonal, so the determinant is obtained by multiplying these values together.

$$\begin{vmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4i \\ -3 & 0 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4i \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= +\begin{vmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4i \end{vmatrix} = (-3)(2i)(-3)(-1)(4i) = 72.$$

Problem 3.13 We need to calculate

$$\begin{vmatrix} 1 & -1 & t-1 \\ 2 & t & -4 \\ 0 & t+2 & -8 \end{vmatrix}$$

and find which values of t, if any, make this determinant 0. We can again use row operations as indicated below.

$$\begin{vmatrix} 1 & -1 & t-1 \\ 2 & t & -4 \\ 0 & t+2 & -8 \end{vmatrix} \underset{R2-2R1}{=} \begin{vmatrix} 1 & -1 & t-1 \\ 0 & t+2 & -2-2t \\ 0 & t+2 & -8 \end{vmatrix}$$

$$\underset{R3-R2}{=}\begin{vmatrix} 1 & -1 & t-1 \\ 0 & t+2 & -2-2t \\ 0 & 0 & 2t-6 \end{vmatrix} = (t+2)(2t-6).$$

The determinant is zero if and only if $t = -2$ or $t = 3$, so these are the two values of t which make the given matrix singular.

Problem 3.14 We know that this sort of transformation multiplies all areas by the absolute value of the determinant of the matrix. (Note that this can lead to notational problems!) With $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ we have $\det(A) = 11$ so all areas will be multiplied by 11.

(a) Since the unit square has area 1, the area of this parallelogram is 11.

(b) Since the unit circle has area π, the ellipse must have area 11π.

Problem 3.15 Given $\mathbf{a} = (a_1, a_2, a_3)$ in $\mathbb{R}^3$, let $\mathbf{x} = (x, y, z)$. Then

$$\mathbf{a} \times \mathbf{x} = (a_2z - a_3y, a_3x - a_1z, a_1y - a_2x) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

So we have

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Problem 3.16 Let A be an invertible $n \times n$ matrix. Suppose, that B is an $n \times n$ matrix such that AB is the $n \times n$ zero matrix Z. We show that B must be the zero matrix Z. To see this, take the equation $AB = Z$ and multiply both sides on the left by A^{-1} to give us $A^{-1}AB = A^{-1}Z$. However, $A^{-1}AB = IB = B$, while $A^{-1}Z = Z$ so we obtain $B = Z$ as required.

4 Linear Systems

A-level revision

Problem 4.1 (a) We have the equations

$$2x - y = 5$$

and

$$x + 3y = 2.$$

Subtracting two of the second equation from the first gives us $-7y = 1$, so $y = -1/7$. Substituting back into either equation then gives us $x = 17/7$.

(b) We are required to find all solutions (if any) to the two simultaneous equations

$$2x - 3y + z = 4$$

and

$$3x + y - 2z = 7.$$

In fact there are infinitely many solutions which form a line in $\mathbb{R}^3$. We need to eliminate one of the variables to get started. Multiplying the first equation by 3 and subtracting 2 of the second equation gives us

$$-11y + 7z = -2.$$

We now give the general solution in terms of z: if $z = t$ then
$y = (-2 - 7z)/(-11) = (2 + 7t)/11$ and (from the first equation)
$x = (4 + 3y - z)/2 = (4 + 6/11 + 21t/11 - t)/2 = 25/11 + (5/11)t$. So the general solution is

$$\begin{aligned}\mathbf{x} = (x, y, z) &= (25/11 + (5/11)t, 2/11 + (7/11)t, t)\\ &= (25/11, 2/11, 0) + t(5/11, 7/11, 1).\end{aligned}$$

(Alternatively, if you see the fraction $1/11$ coming you can set $z = 11t$ to eliminate some of the fractions.)

Module problems

Problem 4.2 The system of equations is

$$x - 2y = 4$$

$$2x + y = 3.$$

This is the same as

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

so we have

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Using our standard formula for inverting 2×2 matrices (for example) we obtain $A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$. The (unique) solution is thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

i.e. $x = 2$, $y = -1$. We then check our answer: $x - 2y = 2 - 2(-1) = 4$ and $2x + y = 4 - 1 = 3$, as required.

Problem 4.3 The system of equations is

$$x_1 - 3x_2 + 2x_4 = 5,$$

$$-2x_2 + x_3 - 4x_4 = 10,$$

$$-3x_1 - 2x_3 + 7x_4 = -2.$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -2 & 1 & -4 \\ -3 & 0 & -2 & 7 \end{bmatrix},$$

the constant vector is $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ -2 \end{bmatrix}$ and the augmented matrix (which may be written with or without the vertical line) is

$$[A\,|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -3 & 0 & 2 & 5 \\ 0 & -2 & 1 & -4 & 10 \\ -3 & 0 & -2 & 7 & -2 \end{array}\right].$$

Problem 4.4 We are given the echelon-form augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 2 & 3 & 6 \end{array}\right]$$

corresponding to a system of 2 equations in 4 unknowns.

(a) We can use, for example, variables x, y, z and t or x_1, x_2, x_3 and x_4. Here we take the first approach to obtain two equations in four unknowns

$$x + 2y - z + 3t = 1,$$

$$2z + 3t = 6.$$

(b) The two leading entries are the 1 in row 1, column 1 and the 2 in row 2, column 3. This means that the pivot columns are columns 1 and 3 and the leading variables (corresponding to these columns) are x and z. The free variables are the other variables, namely y and t.

(c) The solution set is found by noting that the free variables can take arbitrary values, and the leading variables are then found in terms of the free variables. If $y = s$ (and $t = t$) we obtain, from the second equation, $z = (6 - 3t)/2 = 3 - (3/2)t$. From the first equation we obtain $x = 1 - 2y + z - 3t = 1 - 2s + 3 - (3/2)t - 3t = 4 - 2s - (9/2)t$. The general solution is thus

$$(x, y, z, t) = (4-2s-(9/2)t, s, 3-(3/2)t, t) = (4, 0, 3, 0)+s(-2, 1, 0, 0)+t(-9/2, 0, -3/2, 1).$$

Problem 4.5 (a) Our system of equations is

$$x_1 + 2x_2 - x_3 + x_4 = 3$$

$$-x_1 + x_2 + 2x_3 + 3x_4 = 4$$

$$x_1 + 5x_2 + 5x_4 = 10.$$

This corresponds to the following augmented matrix (we omit the vertical line from now on):

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -1 & 1 & 2 & 3 & 4 \\ 1 & 5 & 0 & 5 & 10 \end{bmatrix}.$$

We use row operations to reduce this to echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -1 & 1 & 2 & 3 & 4 \\ 1 & 5 & 0 & 5 & 10 \end{bmatrix} \begin{matrix} \sim \\ R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & 3 & 1 & 4 & 7 \\ 0 & 3 & 1 & 4 & 7 \end{bmatrix} \begin{matrix} \\ \sim \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & 3 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is now in echelon form, and we see that the set of equations is consistent, with pivot columns 1 and 2, leading variables x_1 and x_2 and free variables x_3 and x_4. To find the general solution, we set $x_3 = s$ and $x_4 = t$ (arbitrary) and use back-substitution. We have (from the second row of the echelon matrix) $3x_2 + x_3 + 4x_4 = 7$, so $x_2 = (1/3)(7 - s - 4t) = 7/3 - (1/3)s - (4/3)t$. Then the first equation gives us $x_1 = 3 - 2x_2 + x_3 - x_4 = 3 - 14/3 + (2/3)s + (8/3)t + s - t = -5/3 + (5/3)s + (5/3)t$. The general solution is thus

$$(x_1, x_2, x_3, x_4) = (-5/3 + (5/3)s + (5/3)t, 7/3 - (1/3)s - (4/3)t, s, t)$$

$$= (-5/3, 7/3, 0, 0) + s(5/3, -1/3, 1, 0) + t(5/3, -4/3, 0, 1).$$

(b) This time the equations turn out to be inconsistent. The equations are

$$x + 2y - z = 4$$

$$2x + y + z = 6$$

$$5x + 4y + z = 17.$$

This gives us the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 1 & 1 & 6 \\ 5 & 4 & 1 & 17 \end{bmatrix}.$$

We reduce this to echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 1 & 1 & 6 \\ 5 & 4 & 1 & 17 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -3 & 3 & -2 \\ 0 & -6 & 6 & -3 \end{bmatrix}$$

$$\underset{R_3-2R_2}{\sim}\begin{bmatrix}1 & 2 & -1 & 4\\ 0 & -3 & 3 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

This is now in echelon form: the last row gives us the equation $0 = 1$, so the system is inconsistent.

Problem 4.6 Using $\sharp$ to denote entries that are non-zero and $*$ to denote entries which are arbitrary, the possible echelon forms are

$$\begin{bmatrix}\sharp & * & *\\ 0 & * & *\end{bmatrix},\quad \begin{bmatrix}0 & \sharp & *\\ 0 & 0 & *\end{bmatrix},\quad \text{and}\quad \begin{bmatrix}0 & 0 & *\\ 0 & 0 & 0\end{bmatrix}.$$

The possible reduced echelon forms are

$$\begin{bmatrix}1 & 0 & *\\ 0 & 1 & *\end{bmatrix},\quad \begin{bmatrix}1 & * & 0\\ 0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & * & *\\ 0 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 1 & *\\ 0 & 0 & 0\end{bmatrix},\quad \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\quad \text{and}\quad \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Problem 4.7 The system of equations depends on k, so if we change k we change the solution set. The equations are

$$\begin{aligned} x + 2y + z &= 0\\ 2x + (5+k)y + 6z &= 0\\ 3x + (7+k)y + (11+k)z &= 0. \end{aligned}$$

Since this is a homogeneous system of equations, it can not be inconsistent: we must have either a unique solution or infinitely many solutions. We now form the augmented matrix. We may omit (but not forget) the last column which consists only of zeros. This gives us the matrix

$$A = \begin{bmatrix}1 & 2 & 1\\ 2 & 5+k & 6\\ 3 & 7+k & 11+k\end{bmatrix}.$$

Row reduction gives us

$$\begin{bmatrix}1 & 2 & 1\\ 2 & 5+k & 6\\ 3 & 7+k & 11+k\end{bmatrix} \underset{\substack{R_2-2R_1\\ R_3-3R_1}}{\sim} \begin{bmatrix}1 & 2 & 1\\ 0 & 1+k & 4\\ 0 & 1+k & 8+k\end{bmatrix} \underset{R_3-R_2}{\sim} \begin{bmatrix}1 & 2 & 1\\ 0 & 1+k & 4\\ 0 & 0 & 4+k\end{bmatrix}.$$

This appears to be in echelon form, and certainly it is if $k \neq -1$. Remembering the omitted last column of zeros, we see that if $k \neq -1$ and $k \neq -4$ then we have the unique solution $x = y = z = 0$. If $k = -4$ then we have infinitely many solutions, because z is a free variable. If $k = -1$ the matrix is not yet in echelon form. We have

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 4\\ 0 & 0 & 3\end{bmatrix} \underset{R_3-\frac{3}{4}R_2}{\sim} \begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix},$$

which is now in echelon form. This time y is a free variable, and we have infinitely many solutions. So the answers are (i) never, (ii) $k \neq -1$ and $k \neq -4$, (iii) $k = -1$ or $k = -4$. (An alternative, slightly faster, method uses the theory relating to determinants.)

Problem 4.8 (a) The equations are

$$3x_1 + x_2 = -1,$$
$$4x_1 - x_2 = -6.$$

Modifying the coefficient matrix by replacing the appropriate columns by the constant vector, we have

$$x_1 = \frac{\begin{vmatrix} -1 & 1 \\ -6 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}} = \frac{7}{-7} = -1$$

and

$$x_2 = \frac{\begin{vmatrix} 3 & -1 \\ 4 & -6 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}} = \frac{-14}{-7} = 2.$$

We check this: $3(-1) + 2 = -1$ and $4(-1) - 2 = -6$, as required.

(b) The equations are

$$x_1 + 2x_2 - x_3 = -1,$$
$$2x_1 - 3x_2 + x_3 = 8,$$
$$x_1 - 2x_2 + 3x_3 = 7.$$

This time Cramer's rule gives us

$$x_1 = \frac{\begin{vmatrix} -1 & 2 & -1 \\ 8 & -3 & 1 \\ 7 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{vmatrix}},$$

$$x_2 = \frac{\begin{vmatrix} 1 & -1 & -1 \\ 2 & 8 & 1 \\ 1 & 7 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{vmatrix}}$$

and

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -3 & 8 \\ 1 & -2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{vmatrix}}.$$

If you work these out you obtain the answers $x_1 = 2$, $x_2 = -1$ and $x_3 = 1$, and you may check that this is the correct solution.

Problem 4.9 We have $A = \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}$ and so $\det(A) = -13$. The formula from Chapter 3 gives us immediately

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A) = \frac{1}{-13}\begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1/13 & 4/13 \\ 3/13 & -1/13 \end{bmatrix}.$$

For Gauss-Jordan inversion we start with A on the left and the identity matrix on the right and reduce to reduced echelon form:

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -13 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 13 & 52 & 13 & 0 \\ 0 & -13 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 13 & 0 & 1 & 4 \\ 0 & -13 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/13 & 4/13 \\ 0 & 1 & 3/13 & -1/13 \end{bmatrix}.$$

We now read off the inverse matrix on the right, which is $\begin{bmatrix} 1/13 & 4/13 \\ 3/13 & -1/13 \end{bmatrix}$, as before.

(Clearly the easy formula for the inverse of a 2×2 matrix gives the quicker method here.)

For $B = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 2 & 1 \\ -1 & 1 & 5 \end{bmatrix}$ the two methods are of comparable length and give the answer

$$B^{-1} = \begin{bmatrix} -9 & -17 & 7 \\ -4 & -7 & 3 \\ -1 & -2 & 1 \end{bmatrix}.$$

For larger matrices the Gauss-Jordan method is usually faster than the $\det(A)$, $\mathrm{Adj}(A)$ approach.

Problem 4.10 Let A be a singular $n \times n$ matrix. Then there exists a non-zero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$. Form B by specifying all n of its columns to be this non-zero vector $\mathbf{x}$. Then B is not the zero matrix, but $AB = Z$.

5 Eigenvalues and eigenvectors

Module problems

Problem 5.1 (a) The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix} = 0.$$

The row operation $R1 \to 2R1 - R3$ gives us

$$\begin{vmatrix} 2-2\lambda & 0 & -1+\lambda \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix} = 0.$$

We may remove a factor $(1-\lambda)$ from $R1$ to obtain

$$(1-\lambda)\begin{vmatrix} 2 & 0 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix} = 0.$$

Then the column operation $C1 \to C1 + 2C3$ gives us

$$(1-\lambda)\begin{vmatrix} 0 & 0 & -1 \\ -2 & -\lambda & 5 \\ 2-2\lambda & -2 & -1-\lambda \end{vmatrix} = 0.$$

Expanding by $R1$, we obtain:

$$-(1-\lambda)(4+\lambda(2-2\lambda)) = 0$$

and simplifying and factorizing we obtain

$$(\lambda-1)(-\lambda+2)(\lambda+1) = 0.$$

The eigenvalues are the solutions of this characteristic equation, which are $\lambda = 1, -1$ or 2.

$\underline{\lambda = 1}$

Recall that solving $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to solving $(A-\lambda I)\mathbf{v} = \mathbf{0}$. So we subtract λ (in this case 1) from the diagonal to obtain the matrix $\begin{bmatrix} 2 & -1 & -1 \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{bmatrix}$, and the eigenvalues corresponding to the eigenvalue 1 are then the **non-zero** vectors $\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ solving

$$\begin{bmatrix} 2 & -1 & -1 \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives us a homogeneous system of linear equations in the three variables α, β and γ,

$$2\alpha - \beta - \gamma = 0$$

$$-12\alpha - \beta + 5\gamma = 0$$

$$4\alpha - 2\beta - 2\gamma = 0.$$

We find the general solution in the usual way (it does not really matter what names we give to the variables).

Note that if at this point we can not find a non-zero solution vector, then we must have made a mistake when finding this eigenvalue. The zero vector does **not** count as an eigenvector.)

The general eigenvector here is $\mathbf{v} = \beta \begin{bmatrix} -3 \\ 1 \\ -7 \end{bmatrix}$ for any $\beta \neq 0$. For example, you can check that

$$\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -7 \end{bmatrix}.$$

$\underline{\lambda = -1}$

Again we subtract λ from the diagonal of the original matrix to obtain the matrix

$$\begin{bmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix}.$$

The eigenvectors are then the non-zero vectors solving

$$\begin{bmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this homogeneous system gives us that the general eigenvector is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ for any $\alpha \neq 0$.

For example, you may check that

$$\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$\underline{\lambda = 2}$

This time we subtract two from the diagonal and solve

$$\begin{bmatrix} 1 & -1 & -1 \\ -12 & -2 & 5 \\ 4 & -2 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We then find that the general eigenvector associated with eigenvalue 2 is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ for any $\alpha \neq 0$. In particular, you may check that

$$\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(b) This time the characteristic equation is

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & -1-\lambda & 2 \\ 8 & 3 & -4-\lambda \end{vmatrix} = 0.$$

We may expand directly by $R1$ to obtain

$$(4-\lambda)((-1-\lambda)(-4-\lambda)-6)+8(1+\lambda)=0.$$

This simplifies to

$$-\lambda^3 - \lambda^2 + 30\lambda = 0,$$

and this factorizes, giving,

$$-\lambda(\lambda+6)(\lambda-5) = 0.$$

Thus the eigenvalues are $\lambda = 0, 5 \text{ or } -6$.
NOTE that this time 0 is an eigenvalue. This is not a mistake! In fact every singular square matrix A has 0 as an eigenvalue. It is the eigen**vectors** which are not allowed to be the zero **vector**.

$\underline{\lambda = 0}$
We proceed to subtract λ off the diagonal, as in part (a). Here $\lambda = 0$ so there is no change. We find the non-zero vectors solving

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 8 & 3 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The general eigenvector here is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ -8 \\ -4 \end{bmatrix}$ for $\alpha \neq 0$.

$\underline{\lambda = 5}$
This time we subtract 5 off the diagonal and proceed as before. The general eigenvector corresponding to eigenvalue 5 turns out to be $\mathbf{v} = \beta \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ for $\beta \neq 0$.

$\underline{\lambda = -6}$
Similarly we find that the general eigenvector associated with the eigenvalue -6 is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ 4 \\ -10 \end{bmatrix}$ where $\alpha \neq 0$.

(c) For this part the eigenvalues turn out to be complex. This means that we will be working with complex vectors and scalars. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

This gives us $(1-\lambda)^2 + 1 = 0$, so $\lambda = 1 \pm i$.

$\underline{\lambda = 1+i}$
Subtracting λ from the diagonal gives us the 2×2 matrix $\begin{bmatrix} -i & -\frac{1}{2} \\ 2 & -i \end{bmatrix}$ and we need to solve

$$\begin{bmatrix} -i & -\frac{1}{2} \\ 2 & -i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

You can do this systematically, or, as always with 2×2 matrices, it is easy to spot that one solution is $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. The general eigenvector corresponding to the eigenvalue $1+i$ is any non-zero multiple of this (complex multiples are allowed).

$\underline{\lambda = 1-i}$
Proceeding as above we find that the general eigenvector associated with the eigenvalue $1-i$ is

$$\mathbf{v} = \alpha \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

for any non-zero complex number α.

(d) The characteristic equation is

$$\begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix} = 0.$$

We can spot a factor quickly if we use the column operation $C2 \to C2 + C3$ to obtain

$$\begin{vmatrix} -3-\lambda & 0 & -1 \\ -7 & 4-\lambda & -1 \\ -6 & 4-\lambda & -2-\lambda \end{vmatrix} = 0.$$

We may remove the factor $(4-\lambda)$ from $C2$ to give

$$(4-\lambda)\begin{vmatrix} -3-\lambda & 0 & -1 \\ -7 & 1 & -1 \\ -6 & 1 & -2-\lambda \end{vmatrix} = 0.$$

The row operation $R2 \rightarrow R2 - R3$ simplifies this to

$$(4-\lambda)\begin{vmatrix} -3-\lambda & 0 & -1 \\ -1 & 0 & 1+\lambda \\ -6 & 1 & -2-\lambda \end{vmatrix} = 0.$$

Expanding by $C2$ gives us $-(4-\lambda)((-3-\lambda)(1+\lambda)-1) = 0$ which we first simplify to $(\lambda-4)(-\lambda^2-4\lambda-4) = 0$ and then factorize to obtain $-(\lambda-4)(\lambda+2)^2 = 0$. The eigenvalues are thus $\lambda = 4 \text{ or } -2$ (repeated).

$\underline{\lambda = 4}$

Proceeding as usual, we find that the general eigenvector is $\mathbf{v} = \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, where $\beta \neq 0$.

$\underline{\lambda = -2}$

With repeated eigenvalues you can not be sure 'how many' eigenvectors you will get. You simply proceed to solve the usual system of homogeneous equations and there may or may not turn out to be more than one free variable. In this particular example there is only one free variable. The general eigenvector is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, where $\alpha \neq 0$. We can **not** find two 'independent' eigenvectors corresponding to the eigenvalue -2 in this case.

(e) This is similar to part (d), but this time the repeated eigenvalue **does** give us two 'independent' eigenvectors.

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0.$$

Starting with the column operation $C1 \rightarrow C1 - C3$ and removing a factor $2+\lambda$ from $C1$, we proceed in a similar way to before and find that the characteristic equation is $(2+\lambda)^2(4-\lambda) = 0$

The eigenvalues are thus $\lambda = 4$ or -2 (repeated).

$\underline{\lambda = 4}$
The usual procedure shows that the general eigenvector corresponding to eigenvalue 4 is $\mathbf{v} = \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ where $\alpha \neq 0$.

$\underline{\lambda = -2}$
This time we need to solve the homogeneous system of equations obtained from

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

There is clearly only really one equation here, and we get two free variables. The eigenvectors must still be non-zero, and the general eigenvector can be expressed in parametric form in various ways, for example as

$$\alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where at most one of α and β is allowed to be 0. In particular we can find two 'independent' eigenvectors corresponding to the eigenvalue 2, namely $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. (There will be more on independence later in the module.) There are many other variants on this correct answer.

Problem 5.2 There are many examples. One method is to find all possible 2×2 matrices with the required properties. However it is easier to look at the special case of triangular matrices: we see easily that what we need is a strictly triangular matrix such as $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here the characteristic equation is $\lambda^2 = 0$ and so the only eigenvalue is 0 (repeated).