G12MAN: Mathematical Analysis
Lecture by lecture summary of the module as taught in 2011-12

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Workshop 1: Module information and other documents issued. (All handouts are also available from the web.) Discussion of the nature of the module, and why it is important to attend classes and to try **all** of the question sheets. Revision quiz.

Lecture 1: **Introduction**
Further discussion of the nature of the module and why mathematical analysis and, more generally, rigorous proof/justification are important.
Chapter 1. Introduction to $\mathbb{R}^d$
Revision/notation: sets, subsets, intersections and unions of finitely many sets. Set difference and complement. Cartesian products: definition, notation, examples and non-examples. Some simple sketches of sets.

Lecture 2: Cartesian products of finitely many sets. Revision/notation: $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^d$. Vectors and points. Column vectors. Standard inner product (dot product). Euclidean norm $\|\cdot\|$. Properties of Euclidean norm, including triangle inequality and homogeneity. Euclidean distance $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$ (from Pythagoras). Triangle inequality for Euclidean distance.
Chapter 2. Boundedness of subsets of $\mathbb{R}^d$.
A relevant quote from Shakespeare's Hamlet. Examples of subsets of $\mathbb{R}^d$, especially open balls and closed balls.

Lecture 3: Discussion and sketches of balls in dimensions $d = 1$ and $d = 2$. Examples of bounded sets and unbounded sets. (See also intervals/d-cells below, and many further examples and results on question sheet 1.) Non-empty, bounded sets can be described as sets which have **finite diameter**. Bounded intervals in $\mathbb{R}$. Note in particular alternative notation $]a,b[$ for the open interval with endpoints a and b, to avoid confusion with the ordered pair (a,b). Half-open intervals.

Lecture 4: Most subsets of $\mathbb{R}$ are NOT intervals. The term half-open should **only** be used for (appropriate) intervals or d-cells. Definitions of bounded d-cells in $\mathbb{R}^d$ (pre-2007-8 these were called **intervals** in $\mathbb{R}^d$): rectangles, cuboids, hyper-cuboids etc. Examples of bounded d-cells in $\mathbb{R}^d$. Unbounded d-cells in $\mathbb{R}^d$. End-of-chapter quiz.

Lecture 5: **Chapter 3. Open subsets of $\mathbb{R}^d$**
Interior points, non-interior points and interior of subsets of $\mathbb{R}^d$: definitions and examples. Careful justification of statements concerning interior and non-interior points.

Lecture 6: More examples considered, including $\mathbb{Q}$ and $\mathbb{Q}^c$. Open sets. Definitions of **open** and **not open** in terms of interior points/ non-interior points. Intervals in $\mathbb{R}$: open intervals are open, other types of non-empty interval are not open. Open balls are open. Closed balls are not open. Both the empty set $\emptyset$ and $\mathbb{R}^d$ are open in $\mathbb{R}^d$. Finite sets (sets with only finitely many points in them): **non-empty** finite subsets of $\mathbb{R}^d$ are never open.

Lecture 7: More examples and non-examples of open subsets of $\mathbb{R}^d$. Of the **non-empty**, bounded d-cells (intervals, rectangles, cuboids etc.) in $\mathbb{R}^d$ discussed earlier, the only ones which are open are

(unsurprisingly) the ones that we called 'open d-cells' before. End-of-chapter quiz.

Chapter 4. Topology of $\mathbb{R}^d$

Obvious but important fact: If $A \subseteq B$ then $\operatorname{int} A \subseteq \operatorname{int} B$, i.e., every interior point of A is also an interior point of B.

A union of two open sets is open. An intersection of two open sets is again an open set. An easy induction now shows that finite unions and finite intersections of open sets are still open sets.

Lecture 8: Countable unions (unions of infinite sequences of sets). Countable unions of open sets are open. [So are uncountable unions! See G13MTS for more details.] Countable intersections of sets discussed. An example of a countable intersection of open sets which is not open. Closed sets: a set is closed if its complement is open.

Make sure that you know the difference between 'closed' and 'not open'!

Examples of closed sets. Sets which are both open and closed (clopen sets). There are only two clopen subsets of $\mathbb{R}^d$: they are $\emptyset$ and $\mathbb{R}^d$. Sets which are neither open nor closed.

Lecture 9: Because of de Morgan's laws, countable intersections of closed sets are closed and finite unions of closed sets are closed. Unions of infinitely many closed sets need not be closed.
End-of-chapter quiz.

Chapter 5. Sequences in $\mathbb{R}^d$.

Sequence notation. Convergence of sequences in $\mathbb{R}^d$: definition, terminology and notation. Non-standard terminology: sets which absorb sequences, and the stages by which they absorb the sequences. Examples and exercises involving absorption. The d sequences of coordinates associated with a sequence of vectors in $\mathbb{R}^d$: notation $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,d})$. Interpretation as an array with infinitely many rows and d columns: the rows are the vectors in the sequence, the columns are the sequences in $\mathbb{R}$ corresponding to the individual coordinates. Convergence in terms of convergence of these d individual coordinate sequences: 'coordinatewise convergence'.

Lecture 10: Example of the use of coordinatewise convergence. Recalled: Sandwich Theorem in $\mathbb{R}$. Algebra of limits for sequences in $\mathbb{R}^d$, including the continuity of the norm function from $\mathbb{R}^d$ to $\mathbb{R}$ (see Chapter 7 for full discussion of continuity). Brief discussion of the sequence criterion for closedness: informally this says that a set C is closed if and only if you can not 'escape' from C by taking the limit of a sequence of points from C. Example of a sequence in a subset of $\mathbb{R}$ such that the sequence converges in $\mathbb{R}$ to a limit which is outside the subset.

Lecture 11: Proof of the sequence criterion for closedness. End-of-chapter quiz.

Lecture 12: **Chapter 6. Subsequences and Sequential Compactness**

The nested intervals principle in $\mathbb{R}$ (recalled). The definition of diameter for non-empty, bounded sets in $\mathbb{R}^d$. Examples. The nested d-cells principle in $\mathbb{R}^d$ (nested intervals in d dimensions). Discussion and sketches. Subsequences of sequences: definitions and examples.
Bolzano–Weierstrass theorem stated: every bounded sequence in $\mathbb{R}^d$ has at least one convergent subsequence. Examples.

Lecture 13: Definition and discussion of sequential compactness. Statement and proof of the Heine–Borel Theorem (sequential compactness version), which characterizes the sequentially compact subsets of $\mathbb{R}^d$ as those which are both closed and bounded. Examples of sequentially compact sets. See G13MTS (or books) for the 'topological' notion of compactness in terms of coverings by open sets. (That notion coincides with sequential compactness when dealing with subsets of $\mathbb{R}^d$.)
End-of-chapter quiz.

Lecture 14: **Chapter 7. Functions, limits and continuity**
Functions: revision of domain, co-domain, restriction and image of functions. Functions from $\mathbb{R}^d$ to $\mathbb{R}^l$. Functions from D to $\mathbb{R}^l$ (where $D \subseteq \mathbb{R}^d$). Every function f taking values in $\mathbb{R}^l$ has l corresponding functions taking values in $\mathbb{R}$, one for each coordinate:

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_l(\mathbf{x}))\,.$$

Standard properties of last year's standard functions from $\mathbb{R}$ to $\mathbb{R}$ will be assumed without further proof in this module. In particular, you may state without proof that standard combinations of standard functions from G11CAL are continuous on their domains of definition. (See below for discussion of continuity.)
Limits and continuity of functions: definitions in terms of sequences, recalled for functions from $\mathbb{R}$ to $\mathbb{R}$ and generalized to functions defined on subsets of $\mathbb{R}^d$ and taking values in $\mathbb{R}^l$. Problems with isolated points. Coordinate projections.

Lecture 15: Different ways to approach a point of $\mathbb{R}^2$, and corresponding ways to demonstrate that a function is discontinuous or a function limit does not exist. Properties of some examples of functions from $\mathbb{R}^2$ to $\mathbb{R}$. Comparison of continuity (also called 'joint continuity') and separate continuity. Separate continuity does NOT imply continuity. Example using different curves of approach to show that a function limit does not exist. For $D \subseteq \mathbb{R}^d$, a function f from D to $\mathbb{R}^l$ is continuous if and only if all of the l functions $f_1, f_2, \ldots, f_l$ (corresponding to the l coordinates of $f(\mathbf{x})$, as above) are continuous.
Chapter 8. Further theory of function limits and continuity
New functions from old (pointwise addition, etc.). Algebra of limits for real-valued function limits: deduction from definitions, using the algebra of limits for sequences in $\mathbb{R}$. Sandwich theorem for real-valued function limits.

Lecture 16: Application of the sandwich theorem to showing that a function limit exists for a specific function, using careful case-by-case analysis to estimate the modulus. New continuous functions from old. Pointwise operations for real-valued continuous functions: addition, scalar multiplication, product, modulus, quotient (avoiding division by 0). Composition of continuous functions. Equivalent definitions of function limits and continuity in terms of ε and δ: initial discussion.

Lecture 17: Application of the ε-δ characterization of continuity: continuous real-valued functions that are positive at a point remain positive near the point. Images of sets under continuous functions (continuous images of sets). The continuous image of a sequentially compact set is sequentially compact. Bad behaviour discussed of continuous images of various other types of sets (open sets, closed sets, bounded sets). Brief discussion of the remaining results of the chapter.

Lecture 18: Every non-empty sequentially compact subset of $\mathbb{R}$ has both a maximum and minimum element. The boundedness theorem for continuous real-valued functions defined on non-empty, sequentially compact subsets of $\mathbb{R}^d$. End-of-chapter quiz on Chapters 7 and 8.
Chapter 9. Sequences of functions
Motivation: discussion of applications and various notions of convergence. Recalled notion of absorption. Example illustrating pointwise convergence.
Student Evaluation of Teaching (SET) forms issued.

Lecture 19: Pointwise convergence: definition and examples. Function balls: definition and illustrative sketch. Definition of uniform convergence in terms of function balls and absorption of sequences of functions. Examples illustrating the differences between pointwise and uniform convergence. Uniform convergence is better than pointwise convergence: a uniform limit of continuous functions must be continuous, while a pointwise limit of continuous functions may be discontinuous.

Lecture 20: More examples and theory. Remember that uniform convergence implies pointwise convergence, but **not** conversely. It is impossible for a sequence of unbounded functions to converge uniformly to a bounded function, and it is impossible for a sequence of bounded functions to converge uniformly to an unbounded function. End-of-chapter quiz.

Chapter 10. Rigorous differential calculus.

Definition of differentiability for real-valued functions on open subsets of $\mathbb{R}$. Standard functions from G11CAL have the derivatives given in that module, which you can quote this year. Some functions have problems with their derivatives. For example, $|x|$, $\sqrt{|x|}$ are not differentiable at 0, and also $\arcsin x$ has one-sided infinite gradients at $x = \pm 1$. Similarly $\sqrt{x}$ has a one-sided infinite gradient at $x = 0$. (One can consider one-sided derivatives at endpoints for functions defined on closed intervals, but we will not do so in this module.)

Lecture 21: Differentiability implies continuity, but the converse is untrue as shown by the functions $\sqrt{|x|}$ and $|x|$ (for example). Discussion of the example $f(x) = |x|$. Weierstrass's example of a function which is continuous everywhere but differentiable nowhere on $\mathbb{R}$. Brief discussion/revision of the standard rules for differentiation from G11CAL (product rule, quotient rule, chain rule). See books for rigorous proofs (the chain rule needs care). **From now on, you may assume that all the standard differentiable functions (polynomials, exponential, logarithmic, trigonometric, etc.) you met in G11CAL have the derivatives and properties discussed there.** Fermat's Theorem and Rolle's Theorem, both proved in full. The Mean Value Theorem (MVT): statement and diagram discussed, with a brief discussion of the method of proof.

Lecture 22: Proof of the MVT. Applications of the MVT to prove monotonicity of functions and estimates of function values. End-of-chapter quiz.

Chapter 11. An introduction to Riemann integration

See the slides for details on examinability of material. Motivation: antiderivatives (also called indefinite integrals or primitives) and area. An example of a function which can not be antidifferentiated.

Lecture 23: Partitions. Riemann lower and upper sums (approximation using rectangles). Riemann lower and upper integrals. Riemann integrable functions. Examples of functions which are/are not Riemann integrable. Continuous functions are Riemann integrable. Some discontinuous functions are Riemann integrable. Standard properties of the Riemann integral. The (first) Fundamental Theorem of Calculus and its applications.