8 Further theory of function limits and continuity

8.1 Algebra of limits and sandwich theorem for real-valued function limits

The following results give versions of the algebra of limits and sandwich theorem for real-valued functions defined on subsets of $\mathbb{R}^d$.

Don’t forget that there is a big difference between limits of sequences and function limits, even though there are connections between them.

You should ensure that you understand the different nature of these different types of limit.

Suppose that $D \subseteq \mathbb{R}^d$, $\lambda \in \mathbb{R}$, and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ are functions.

We may form new functions in the usual way (using what are often described as pointwise operations, e.g. pointwise addition, etc.).
\[(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x),\]

\[(\lambda f)(x) = \lambda f(x) \quad \text{and} \quad |f|(x) = |f(x)|,\]

where \(x \in D\).

These new functions also have domain \(D\).

The pointwise product \(fg\) should not be confused with the composite of the two functions \(f\) and \(g\).

Finally if \(g(x) \neq 0\) then the quotient

\[(f/g)(x) = \frac{f(x)}{g(x)}\]

makes sense and defines a function \(f/g\), at least at those points of \(D\) where \(g\) is non-zero.
Theorem 8.1.1 (Algebra of real-valued function limits) Let $D$ be a non-empty subset of $\mathbb{R}^d$ and suppose that $a$ is a non-isolated point of $D$. Let $f$ and $g$ be functions from $D$ to $\mathbb{R}$ (or from $D \setminus \{a\}$ to $\mathbb{R}$), and let $\lambda \in \mathbb{R}$.

Suppose that $\lim_{x \to a} f(x)$ exists and is $L_1 \in \mathbb{R}$ and that $\lim_{x \to a} g(x)$ exists and is $L_2 \in \mathbb{R}$.

Then

$$\lim_{x \to a} (f(x) + g(x)) = L_1 + L_2,$$

$$\lim_{x \to a} (f(x)g(x)) = L_1 L_2,$$

$$\lim_{x \to a} \vert f(x) \vert = \vert L_1 \vert$$ and

$$\lim_{x \to a} (\lambda f(x)) = \lambda L_1.$$

Moreover, provided that $g$ does not take the value 0 anywhere on $D \setminus \{a\}$, and that $L_2 \neq 0$, we have

$$\lim_{x \to a} (f(x)/g(x)) = L_1/L_2.$$

Gap to fill in

Let $\{x_n\} \subseteq D \setminus \{a\}$ with $x_n \to a$ as $n \to \infty$.

[We are allowed to use alg. of limits for real sequences]
Then, \( f(x_n) \to L_1 \) as \( n \to \infty \) and \( g(x_n) \to L_2 \) as \( n \to \infty \).

Gap to fill in

Because \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} g(x) = L_2 \),

By algebra of limits for real sequences,

\[
\begin{align*}
f(x_n) + g(x_n) & \to L_1 + L_2 \quad \text{as} \quad n \to \infty \\
f(x_n) g(x_n) & \to L_1 L_2 \quad \text{as} \quad n \to \infty
\end{align*}
\]

Since this holds for all such sequences \( (x_n) \), we deduce

\[
\lim_{x \to a} (f(x) + g(x)) = L_1 + L_2
\]

\[
\lim_{x \to a} (f(x) g(x)) = L_1 L_2
\] (et c.)
Theorem 8.1.2 (Sandwich Theorem for real-valued function limits) Let $D$ be a non-empty subset of $\mathbb{R}^d$ and suppose that $a$ is a non-isolated point of $D$. Let $f$, $g$ and $h$ be functions from $D$ to $\mathbb{R}$ (or from $D \setminus \{a\}$ to $\mathbb{R}$), and let $L \in \mathbb{R}$. Suppose that, for all $x \in D \setminus \{a\}$, we have $f(x) \leq g(x) \leq h(x)$, and that both $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$. Then $\lim_{x \to a} g(x)$ also exists, and is equal to $L$.

The proof of this is an exercise.

Gap to fill in
Corollary 8.1.3 Let $D$ be a non-empty subset of $\mathbb{R}^d$ and suppose that $a$ is a non-isolated point of $D$. Let $f$ be a function from $D$ to $\mathbb{R}$ (or from $D \setminus \{a\}$ to $\mathbb{R}$). Then

$$\lim_{x \to a} |f(x)| = 0 \iff \lim_{x \to a} f(x) = 0.$$  

Gap to fill in

$$(\Leftarrow)$$ is from algebra of limits.

$$(\Rightarrow)$$ Given $|f(x)| \to 0$ as $x \to a$, then also have $-|f(x)| \to 0$ as $x \to a$ (by algebra of limits). Then

$$-|f(x)| \leq f(x) \leq |f(x)|$$

So, by sandwich theorem, since both $-|f(x)| \to 0$ and $|f(x)| \to 0$. 

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Example. Prove that

\[
\lim_{(x,y) \to (0,0)} \left( \frac{xy^3 \sin y}{x^2 + y^6} \right) = 0.
\]

Gap to fill in

We use sandwich theorem and case by case analysis.

Case 1. Find a quick proof using AM/GM inequality.

Case by case analysis.

First, using our rule of thumb (to find the “decisive curve”) we check the curve where \( x = y^3 \), making terms in denominator equal. In chapter 7, this helped show limit did not exist.

With \( y \neq 0 \) and \( \frac{1}{y} (\cos y) \neq 0 \) the function
has values \[ \frac{y^3 y^3 \sin y}{y^6 + y^6} = \frac{1}{2} \sin y \]

Gap to fill in

\[ \lim_{y \to 0} f(y) = 0 \]

as \( y \to 0 \).

Set \( f(x, y) = \frac{xy^3 \sin y}{x^2 + y^6} \)

for \((x, y) \neq (0, 0)\).

Estimate \( |f(x, y)| \).

\[ |f(x, y)| \leq |\sin y| \]

[We show \( |f(x, y)| \leq |\sin y| \) and then use sandwich theorem.]

Case I [Based on “dense in curve”]

\[ |x| \leq (y)^3 \]

In this case and \( y \not\to 0 \) and

\[ |f(x, y)| = \left| \frac{xy^3 \sin y}{x^2 + y^6} \right| \]
\[
\frac{|x||y|^3|siny|}{|x|^2 + |y|^6}
\]

Gap to fill in

\[
\leq \frac{|x||y|^3|siny|}{|y|^6}
\]

(smaller)

\[
= \frac{|x||siny|}{|y|^3}
\]

\[
\leq |siny| \quad \text{by Case I assumption.}
\]

Case II \[ |x| \geq |y|^3 \]

In this case \( x \neq 0 \) and

\[
|f(x,y)| = \frac{|x||y|^3|siny|}{|x|^2 + |y|^6}
\]

\[
\leq \frac{|x||y|^3|siny|}{|x|^2}
\]
\[
= \frac{|\gamma|}{|\gamma|} |\sin y|
\leq |\sin y| \quad \text{by case II assumption}
\]

In both cases, \( |f(x,y)| \leq |\sin y| \rightarrow 0 \) as \((x,y) \rightarrow (0,0)\).

So this inequality holds for all \((x,y) \neq (0,0)\).

We have \(\frac{\partial}{\partial y} f(x,y)\) (standard function)

\(0 \leq |f(x,y)| \leq |\sin y| \rightarrow 0 \) as \((x,y) \rightarrow (0,0)\)

By the sandwich theorem, \( |f(x,y)| \rightarrow 0 \) as \((x,y) \rightarrow (0,0)\)

and hence \( f(x,y) \rightarrow 0 \) as \((x,y) \rightarrow (0,0)\). \(\square\)
8.2 New continuous functions from old!

The following results allow us to show that many types of functions are continuous.

For instance, we can use it to show that the two examples we looked at earlier of functions on $\mathbb{R}^2$ which were discontinuous at $(0,0)$ are, in fact, continuous everywhere else in $\mathbb{R}^2$.

**Theorem 8.2.1** Let $\lambda \in \mathbb{R}$. If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both continuous at a point $a \in D$, then the following functions are also continuous at $a$:

$$f + g; \quad \lambda f; \quad fg; \quad |f|.$$

Moreover, if $g$ does not take the value 0 on $D$, then $f/g$ is also continuous at $a$.  

**Gap to fill in**

Just like algebra of limits for function limits.

If $a$ is interior in $D$ then functions from $D \setminus \{a\} \to \mathbb{R}$
are always continuous at \( a \). If not, we can use functions limits, or use sequences:

\[ (x_n) \leq D \text{ and } x_n \to a \text{ as } n \to \infty, \text{ since } f \text{ and } g \text{ are continuous at } a, \text{ we must have} \]

\[ f(x_n) \to f(a) \text{ and } g(x_n) \to g(a) \text{ as } n \to \infty. \]

Using algebra of limits for real sequences:

\[ (f + g)(x_n) = f(x_n) + g(x_n) \]

\[ \frac{x}{x + y} = \frac{a}{a} \]

\[ \rightarrow f(x) + g(x) = (f + g)(x) \]

and

\[ (fg)(x_n) = f(x_n)g(x_n) \]

\[ \rightarrow f(x)g(x) = (fg)(x) \text{ as } n \to \infty. \]
Corollary 8.2.2 Let $\lambda \in \mathbb{R}$.

If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are continuous (i.e. continuous at every point of $D$) then so are $f + g$, $\lambda f$, $fg$ and $|f|$.

Also, the function $\frac{f}{g}$ is continuous, provided that $g$ does not take the value 0 on $D$.

Proof. This is immediate from the preceding theorem.

Another result of this type concerns composition of continuous functions.

Theorem 8.2.3 (Continuity of composite functions)
Let $f : \mathbb{R}^d \to \mathbb{R}^l$ and $g : \mathbb{R}^l \to \mathbb{R}^k$ be two continuous functions, where $d, l, k$ are positive integers.

Then the composite function $g \circ f : \mathbb{R}^d \to \mathbb{R}^k$ defined by $(g \circ f)(x) = g(f(x))$ for $x \in \mathbb{R}^d$ is also continuous.

Proof.

Gap to fill in

Let \((x_n)\) be a sequence in \(\mathbb{R}^d\) with \(\lim_{n \to \infty} x_n = x\),
say. [We must show that
\[(g \circ f)(x_n) \to (g \circ f)(x)\] as \(n \to \infty\).

**Gap to fill in**

Because \(f\) is continuous \(\mathbb{R}^d \to \mathbb{R}^l\) and \(x_n \to x\) in \(\mathbb{R}^d\) as \(n \to \infty\),
\[f(x_n) \to f(x)\] in \(\mathbb{R}^l\) as \(n \to \infty\).

But \(g \circ \sigma_3 \mathbb{R}^l \to \mathbb{R}^k\), so
\[g(f(x_n)) \to g(f(x))\] as \(n \to \infty\).

i.e. \((g \circ f)(x_n) \to (g \circ f)(x)\)

as \(n \to \infty\).

This holds whenever \((x_n)\) is convergent, so \(g \circ f\) is continuous.
\(\square\)

We now know property

that “standard compositions

of its functions are \(C^1\)”.
This result also works point by point, provided that the domains match up.

If $f$ is continuous at $a$ and $g$ is continuous at $f(a)$ then, restricting attention to a domain where $g \circ f$ can be defined, $g \circ f$ is continuous at $a$.

(Exercise. Fill in the details.)

Gap to fill in

\[ \text{Gap to fill in} \]
8.3 Equivalent Definitions of Limits and Continuity (in terms of $\varepsilon$ and $\delta$)

We begin by recalling two results from the module G11ACF.

**Proposition 8.3.1 (Characterization of the limit of $f : \mathbb{R} \to \mathbb{R}$)**

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$.

Then the limit $\lim_{x \to a} f(x)$ of $f(x)$ as $x$ tends to $a$ exists and equals $L \in \mathbb{R}$ if and only if the following condition holds:

for every $\varepsilon > 0$ there is $\delta > 0$ such that, for $x \in \mathbb{R}$, we have

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \quad (1)$$

Note that $\delta$ will usually depend on $\varepsilon$.

The smaller $\varepsilon$ is, the smaller we will need to make $\delta$ in order for (1) to hold.
Correspondingly, for continuity, we have the following result.

**Proposition 8.3.2 (Characterization of continuity of \( f : \mathbb{R} \to \mathbb{R} \) at \( a \))**

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function and \( a \in \mathbb{R} \).

Then \( f \) is continuous at \( a \) if and only if the following condition holds:

for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that, for \( x \in \mathbb{R} \), we have

\[
|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.
\]

Now if \( D \subseteq \mathbb{R}^d \) and \( f : D \to \mathbb{R}^l \) then we have the same pair of propositions for the function \( f \).

The proofs are omitted, as they are essentially identical to those of the results from G11ACF which we recalled above.

*(Exercise. Fill in the details here.)*
Proposition 8.3.3 (Characterization of the limit of \( f : D \to \mathbb{R}^l \))

Let \( f : D \to \mathbb{R}^l \) be a function and suppose that \( a \) is a non-isolated point of \( D \).

Then the limit \( \lim_{x \to a} f(x) \) of \( f(x) \) as \( x \) tends to \( a \) exists and equals \( q \in \mathbb{R}^l \) if and only if the following condition holds:

for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that, for \( x \in D \),

\[
0 < \|x - a\| < \delta \implies \|f(x) - q\| < \varepsilon.
\]

Proposition 8.3.4 (Characterization of continuity of a function \( f : D \to \mathbb{R}^l \) at \( a \in D \)) Let \( f : D \to \mathbb{R}^l \) be a function and \( a \in D \).

Then \( f \) is continuous at \( a \) if and only if the following condition holds:

for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that, for \( x \in D \),

\[
\|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.
\]

Can think of \( \delta \) as a function of \( \varepsilon \). (Maybe \( \delta \ll \varepsilon \))
The following standard fact about real numbers often helps when investigating the implications of continuity.

Let \( a \) and \( b \) be real numbers and let \( \varepsilon > 0 \). Then

\[
a \in ]b - \varepsilon, b + \varepsilon[ \iff b \in ]a - \varepsilon, a + \varepsilon[
\]

\[
\iff |a - b| < \varepsilon \iff -\varepsilon < a - b < \varepsilon
\]

**In particular, if** \(|a - b| < \varepsilon\), **then we have** \(a > b - \varepsilon\) **and** \(b > a - \varepsilon\).

The next lemma, which is often useful, can be shown using the \(\varepsilon\)-\(\delta\) characterization of continuity above.

**Lemma 8.3.5** Let \( f \) be a continuous function from \( \mathbb{R}^d \) to \( \mathbb{R} \). Set

\[
U = \{x \in \mathbb{R}^d \mid f(x) > 0\}.
\]

Then \( U \) is an open subset of \( \mathbb{R}^d \).

**Gap to fill in**

**Proof** (Using \(\varepsilon\)-\(\delta\)).

[We show \(U \subseteq \text{int}(U)\)].

Let \( x \in U \), so that \( f(x) > 0 \).
Set \( \varepsilon = \frac{1}{2} f(x) > 0 \) \( [f(x) \text{ would do}]. \)

Since \( f \) is continuous at \( x, \)

\[ \exists \delta > 0 \text{ s.t. } \forall y, \|y - x\| < \delta \]

we have \( |f(y) - f(x)| < \varepsilon. \)

But then \( f(y) > f(x) - \varepsilon = f(x) - \frac{1}{2} f(x) = \frac{1}{2} f(x) > 0, \)

so \( y \in U. \)

This shows \( B_\delta(x) \subseteq U. \)

So \( x \in \text{int } (U). \)

True for all \( x \in U, \) so \( U \)

is open. \( \square \)

Exercise: Try to prove this using sequences \( (messie) \)
8.4 Images under continuous functions

The following is a very important property of sequentially compact sets, and may be summarized as follows.

**The continuous image of a sequentially compact set is always sequentially compact.**

**Theorem 8.4.1** Let $D$ be a non-empty, sequentially compact subset of $\mathbb{R}^d$, and suppose that $f : D \to \mathbb{R}^l$ is a continuous function.

Then the image of $D$ under $f$, $f(D) = \{f(x) \mid x \in D\}$, is a sequentially compact subset of $\mathbb{R}^l$.

**Proof.**

[Use def. of seq. compact]

[Gap to fill in]

[We must prove $f(D)$ is seq compact, i.e., every sequence in $f(D)$ has at least one subsequence which converges and has limit in $f(D)$.]
Let \((y_n)\) be a sequence in \(f(D)\). Then we can choose \(x_n\) in \(D\) with \(f(x_n) = y_n\) \((n \in \mathbb{N})\).

Then \((x_n) \subseteq D\) and \(D\) is sequentially compact, so \(\exists n_1 < n_2 < n_3 \ldots \) in \(\mathbb{N}\) with \((x_{n_k})\) convergent to some point, say \(x^*\), in \(D\).

Since \(f: D \to \mathbb{R}\) is continuous, we have \(f(x_{n_k}) \to f(x^*)\) as \(k \to \infty\).

Since \(x^* \in D\), \(f(x^*) \in f(D)\).

But \(f(x_{n_k}) = y_{n_k}\) \(\therefore\) \(\{y_{n_k}\}\) converges to \(f(x^*) \in f(D)\).

This shows \((y_n)\) has a convergent subsequence whose limit is in \(f(D)\).
The situation is different for continuous images of other types of sets.

1) Continuous images of open sets need not be open.

Gap to fill in

Find an example of an open set \( U \) and a \( C^1 \) function \( f: U \to \mathbb{R} \) such that \( f(U) \) is not open.

For example, \( f: \mathbb{R} \to \mathbb{R} \)

\[
f(x) = \sin x.
\]

Consider \( U = \mathbb{R} \) (open)

\[
f(U) = [-1, 1]
\]

which is not open.

[Or use constant function, e.g. \( f(x) = 0 \) for all \( x \).]
2) Continuous images of closed sets need not be closed.

**Gap to fill in**

For example, take

\[ f: \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x. \]

\[ y = e^x \]

\[ \mathbb{R} \text{ is closed but } f(\mathbb{R}) = [0, \infty), \text{ which is not closed.} \]
3) Continuous images of bounded sets need not be bounded.

**Gap to fill in**

Set $E = \bigcup_{j=1}^{\infty} \left[ 0, 1 \right]$

and define $f : E \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. (Defined and $\leq$ at all points of $E$.)

Then $f(E) = \bigcup_{j=1}^{\infty} \left[ 1, \infty \right]$. So $E$ is bounded, but $f(E)$ is unbounded.

**Exercise.** Find some more examples for yourself!
The special case of the above theorem where $D \subseteq \mathbb{R}^d$ is sequentially compact and $f : D \to \mathbb{R}$ is continuous is particularly important.

Before looking at this, we prove an important lemma.

**Lemma 8.4.2** Every non-empty, sequentially compact subset of $\mathbb{R}$ has both a maximum element and a minimum element.

**Proof.** Let $E$ be a non-empty seq. compact subset of $\mathbb{R}$.
By H-B Theorem, $E$ is closed and bounded.

Gap to fill in

Since $E$ is non-empty and bounded, $E$ has a least upper bound $\sup E$ and a greatest lower bound $\inf E$ in $\mathbb{R}$.

To finish, prove that $\sup E$ and $\inf E$ are elements of $E$.

We prove for $\sup$ ($\inf$ is similar).

Set $s = \sup E$. Since $s$ is an upper bound for $E$, we have

1. for all $x \in E$, $x \leq s$.

Since $s$ is the least upper bound for $E$, then for all $n \in \mathbb{N}$, $s - \frac{1}{n}$ is not an upper bound for $E$, $\exists x_n \in E$ with $x_n > s - \frac{1}{n}$.

Then $s - \frac{1}{n} < x_n \leq s$. By sandwich theorem $x_n \to s$.
Theorem 8.4.3 (The Boundedness Theorem for sequentially compact subsets of $\mathbb{R}^d$.) Let $D$ be a non-empty, sequentially compact subset of $\mathbb{R}^d$, and let $f$ be a continuous function from $D$ to $\mathbb{R}$.

Then there are $p$ and $q \in D$ such that, for all $x \in D$, we have $f(p) \leq f(x) \leq f(q)$.

**Remark.** Thus, in this setting, $f$ is bounded above and below on $D$ and, among the values $f$ takes on $D$, $f$ attains a maximum value at some point $q$ of $D$ and a minimum value at some point $p$ of $D$.

**Proof.**

Gap to fill in

Since $D$ is seq. compact and $f$ is cont., $f(D)$ is seq. compact in $\mathbb{R}$, and non-empty since $D \neq \emptyset$.

By Lemma, $f(D)$ has a max element $y_{\text{max}}$ and a min element $y_{\text{min}}$.

Choose $p \in D$ with $f(p) = y_{\text{min}}$ and $q \in D$ with $f(q) = y_{\text{max}}$. 

But $x_n \in E$ and $E$ is closed, so $s = \lim_{n \to \infty} x_n \in E$ by seq. criterion. □
Then for all $x \in D$ we have $y_{\text{min}} \leq f(x) \leq y_{\text{max}}$, i.e.

Gap to fill in

$f(x) \leq f(y) \leq f(z)$, as required.
In particular: If \( f : [a, b] \to \mathbb{R} \) is continuous then there are \( p, q \in [a, b] \) such that, for all \( x \in [a, b] \), we have
\[
  f(p) \leq f(x) \leq f(q).
\]
(This is the version of the Boundedness Theorem which you met in G11ACF.)

Of course, \( p, q \) do not depend on \( x \)!
Quiz (Chapters 7 and 8)

Does \( \lim_{(x,y) \to (0,0)} f(x,y) \) exist?

For the following functions \( f \):

If so, what is the limit?

(1) \( f(x,y) = \cos(xy) \)

\( \lim_{(x,y) \to (0,0)} f(x,y) = \cos(0 \times 0) = \cos(0) = 1 \)

(2) \( f(x,y) = \frac{xy^2 e^x}{x^2 + y^4} \)

\( (x, y) \neq (0, 0) \). \text{ Limit does not exist } \( \lim_{(x,y) \to (0,0)} f(x,y) \)

(3) \( f(x,y) = \frac{xy^2 \sin x}{x^2 + y^4} \)

\( (x, y) \neq (0, 0) \). \text{ Limit does not exist } \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \)

[Exc. check details.]