Remember $f_n(x) = n^2 x^n (1-x)$
and $g_n(x) = n^3 x^n (1-x)$ from start of module.

9 Sequences of Functions

"Sequence of curves"

9.1 Pointwise convergence and uniform convergence

Our examples have $d=1$.

Let $D \subseteq \mathbb{R}^d$, and let $f_n : D \to \mathbb{R}$ be functions $(n \in \mathbb{N})$.

We may think of the functions $f_1, f_2, f_3, \ldots$ as forming a **sequence of functions**.

This is very different from a sequence of numbers but it is still possible to define the concept of a limit of such a sequence.

Such a notion is quite important.

For instance when solving differential equations or other problems one is often able to produce a sequence of approximate solutions and then needs to know in which sense the approximate solutions converges to the exact one.

We shall discuss two of the main notions of convergence of sequences of functions $f_n : D \to \mathbb{R}$, **pointwise convergence** and **uniform convergence**.

To help us, recall the following **non-standard terminology**, introduced earlier.

Definition 9.1.1 Let $(\boldsymbol{x}_n) \subseteq \mathbb{R}^d$ and let $A \subseteq \mathbb{R}^d$.

We say that the set A **absorbs** the sequence $(\boldsymbol{x}_n)$ if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } \boldsymbol{x}_n \in A, \qquad (*)$$

i.e., all terms of the sequence from $\boldsymbol{x}_N$ onwards lie in the set A.

The following result appears on the question sheet **More practice with definitions, proofs and examples**.

The proof is an **NEB exercise**.

Proposition 9.1.2 Let $a \in \mathbb{R}$ and let $(x_n) \subseteq \mathbb{R}$. Then the following statements are equivalent:

(a) the sequence (x_n) converges to a;

(b) for all $\varepsilon > 0$, the closed interval $[a - \varepsilon, a + \varepsilon]$ absorbs the sequence (x_n).

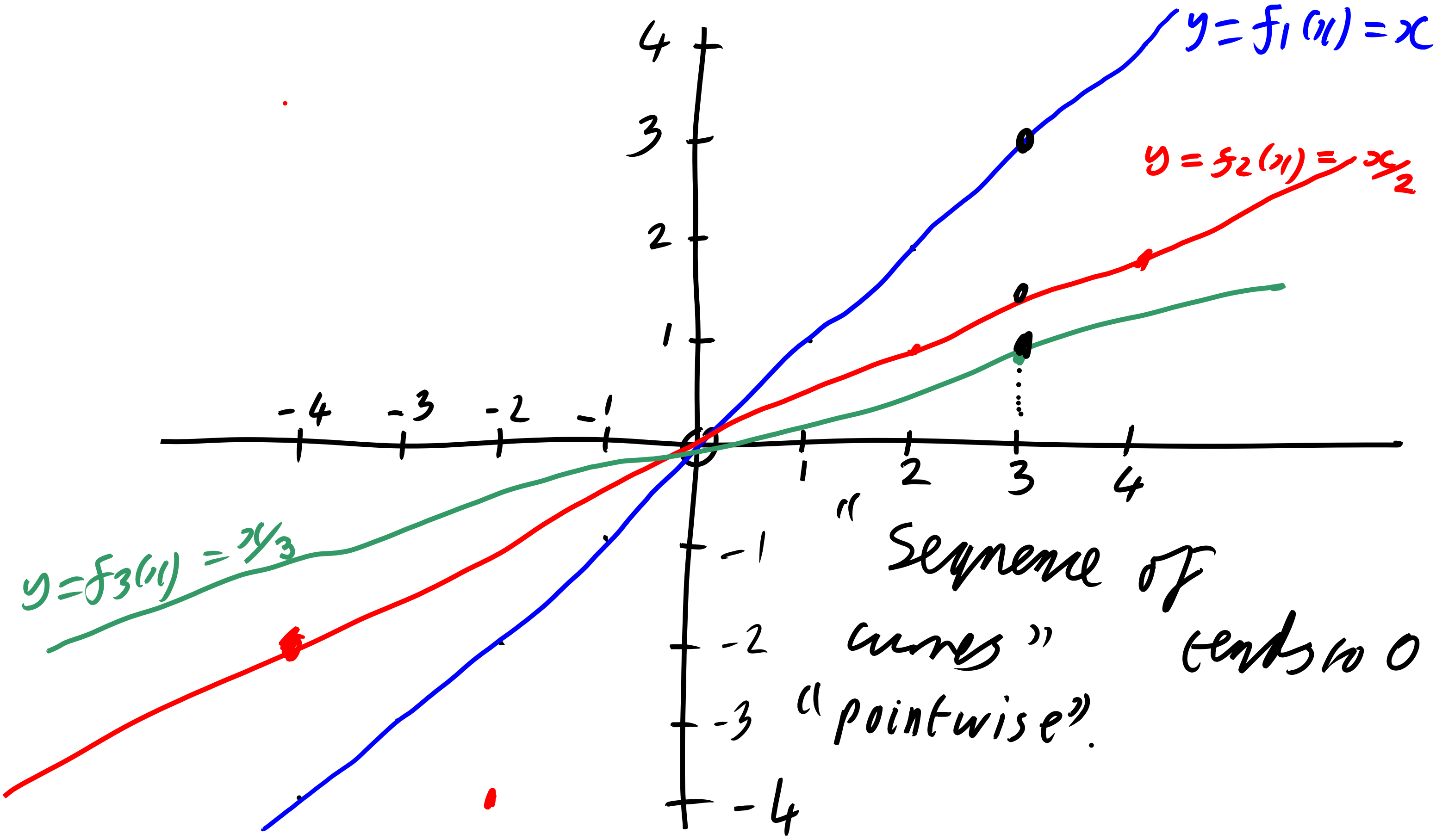
The following example illustrates the notion of **pointwise convergence** for a sequence of functions.

Example 9.1.3 Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x/n$ $(n \in \mathbb{N}, x \in \mathbb{R})$. Sketch the functions f_1, f_2 and f_3 on the same set of axes, and, for all $x \in \mathbb{R}$, determine $\lim_{n\to\infty} f_n(x)$.

Gap to fill in

$f_1(x) = x/1 = x$

$f_2(x) = x/2$, $f_3(x) = x/3$ $(x \in \mathbb{R})$.



For each $x \in \mathbb{R}$, we have

$$f_n(x) = \frac{x}{n} \to 0 \text{ as } n \to \infty.$$

Gap to fill in

[For example, $f_n(3) = \frac{3}{n} \to 0$ as $n \to \infty$.]

This gives POINTWISE CONVERGENCE

to the constant function 0.

Definition 9.1.4 Let $D \subseteq \mathbb{R}^d$, let $f_n : D \to \mathbb{R}$ be functions $(n \in \mathbb{N})$, and let $f : D \to \mathbb{R}$. The sequence of functions (f_n) **converges pointwise (on D)** to the function f if, for every $\boldsymbol{x} \in D$, the sequence $(f_n(\boldsymbol{x}))_{n=1}^{\infty}$ converges to $f(\boldsymbol{x})$, i.e.,

for all $x \in D$, we have $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ as $n \to \infty$.

Gap to fill in

Above, $D = \mathbb{R}$, $f_n(x) = x/n$
$f(x) = 0$ $\quad (x \in \mathbb{R}, n \in \mathbb{N})$.

$f_n(x) = \frac{x}{n} \to 0 = f(x)$ as $n \to \infty$.
True all x in $\mathbb{R}$, so $f_n \to f$ pointwise on $\mathbb{R}$.

The notion of uniform convergence is more subtle.

To explain this, we first extend our notions of closed ball and of sets absorbing sequences.

We need to consider sets and sequences of **functions**.

Definition 9.1.5 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \mapsto \mathbb{R}$, and let (f_n) be a sequence of functions from D to $\mathbb{R}$.

For $\varepsilon > 0$, we define the **closed function ball** (or just **closed ball**) centred on f and with radius ε, $\bar{\boldsymbol{B}}_\varepsilon(f)$, by

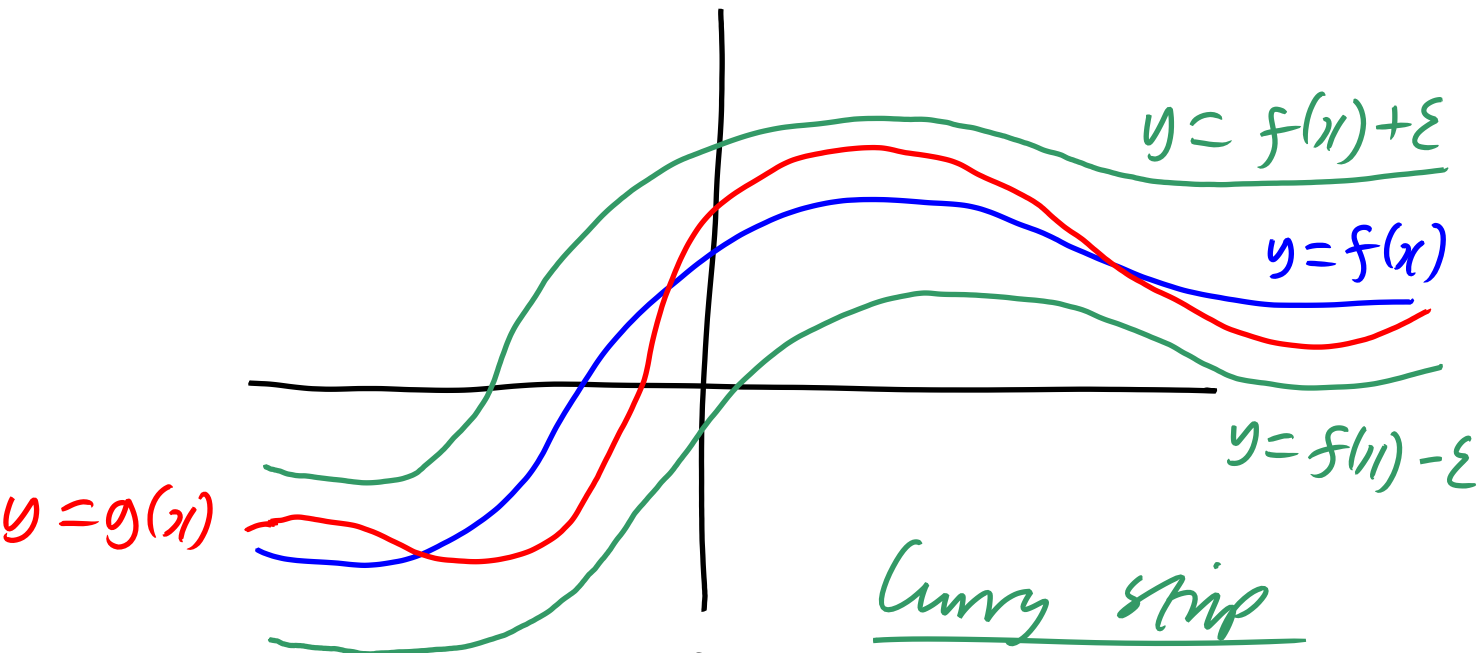
$$\bar{\boldsymbol{B}}_\varepsilon(f) = \{g : D \to \mathbb{R} \mid |g(\boldsymbol{x}) - f(\boldsymbol{x})| \leq \varepsilon \text{ for all } \boldsymbol{x} \in D\}$$

$$= \{g : D \to \mathbb{R} \mid g(\boldsymbol{x}) \in [f(\boldsymbol{x}) - \varepsilon, f(\boldsymbol{x}) + \varepsilon] \text{ for all } \boldsymbol{x} \in D\}.$$

Note that this closed ball is a set of **functions** from D to $\mathbb{R}$. This is why we use the **slightly non-standard** term **closed function ball**.

Gap to fill in

For example, suppose $D = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$.



Maybe $\varepsilon = 10^{-9}$?

Need $f(x)-\varepsilon \leq g(x) \leq f(x)+\varepsilon$ for all x in $\mathbb{R}$ to have $g \in \bar{B}_\varepsilon(f)$.

As in $(*)$ above, we say that the closed function ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ **absorbs** the sequence of functions (f_n) if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } f_n \in \bar{\boldsymbol{B}}_\varepsilon(f), \tag{$**$}$$

i.e., all terms of the sequence of functions from f_N onwards lie in $\bar{\boldsymbol{B}}_\varepsilon(f)$.

> The sequence (f_n) **converges uniformly (on D)** to the function f if, for every $\varepsilon > 0$, the closed function ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ absorbs the sequence (f_n).

In full, this means the following:

> For all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $\boldsymbol{x} \in D$, we have
>
> $$|f_n(x) - f(x)| \leq \varepsilon \, .$$

The N in the full definition of uniform convergence depends only on ε; the same N works for all $\boldsymbol{x} \in D$.

Roughly this means that the sequences of numbers $(f_n(\boldsymbol{x}))$ converge to $f(\boldsymbol{x})$ at the same rate.

Pointwise convergence on the other hand means simply that the sequence of numbers $(f_n(\boldsymbol{x}))$ converges to $f(\boldsymbol{x})$ for each $\boldsymbol{x} \in D$.

At different points the speed of convergence could be very different.

The next result follows directly from the definitions.

(**Exercise.** Convince yourself that this is correct.)

Lemma 9.1.6 If (f_n) converges uniformly to f (on D) then it converges pointwise to f on D.

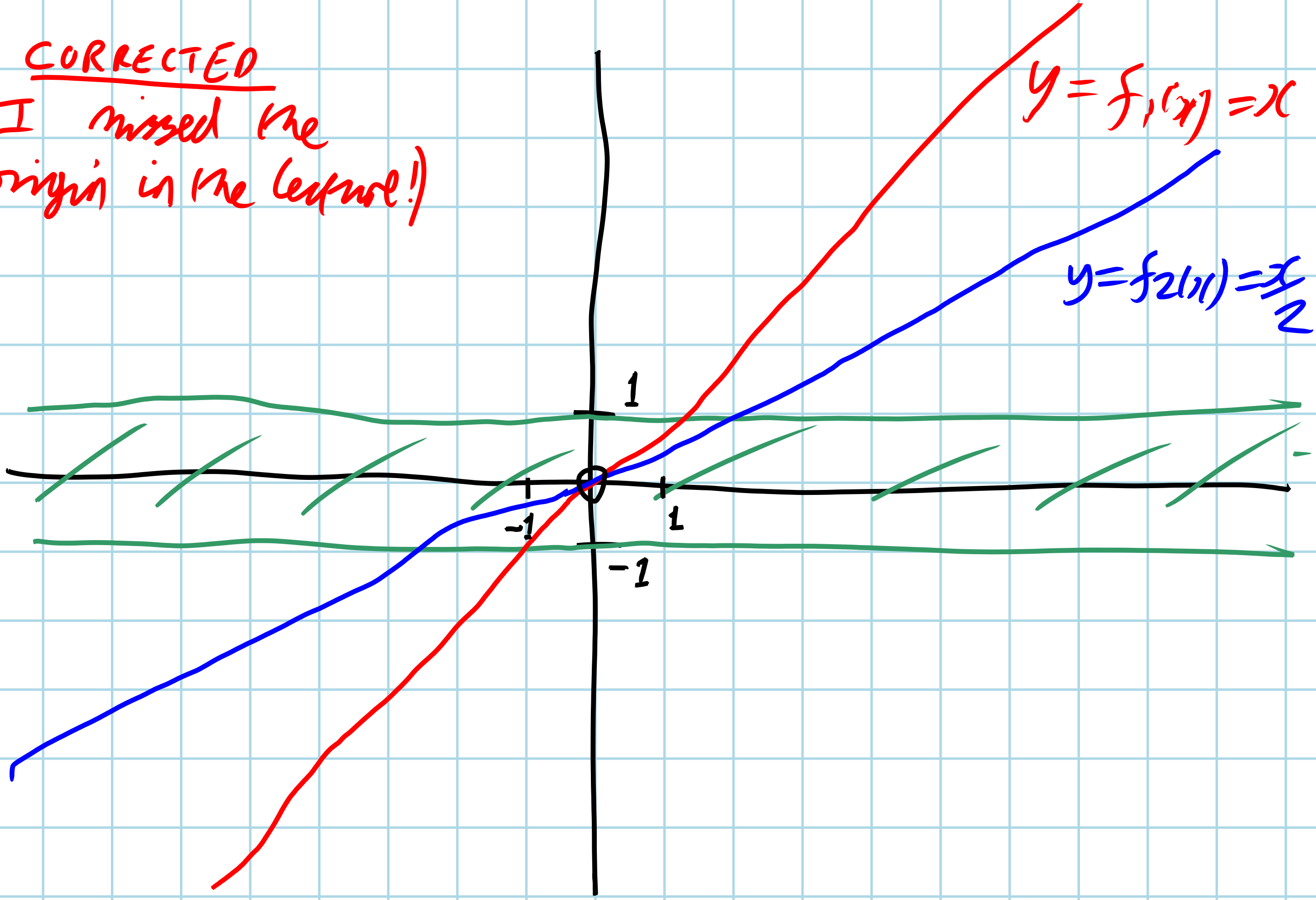
The converse of this lemma is NOT true, as you can see from the functions x/n that we looked at earlier.

Gap to fill in

Only need one bad ε! But for $f_n(x) = x/n$, $f(x) = 0$, all ε are bad. For example, take $\varepsilon = 1$.

$\bar{B}_1(f)$ is horizontal strip between $y=-1$ and $y=1$.

CORRECTED
(I missed the origin in the lecture!)



NONE of the functions

$f_n(x) = x/n$ are in $\overline{B}_1(f)$

Explicitly $f_n(2n) = \frac{2n}{n} = 2 > 1$.

So certainly $\overline{B}_1(f)$ does not absorb (f_n).

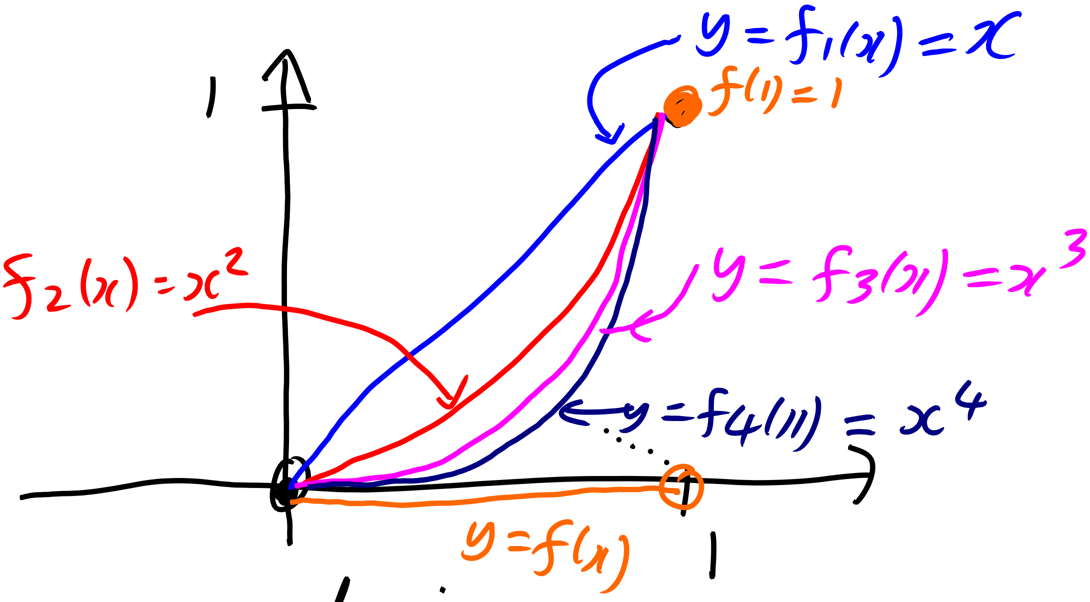
[A bit more work shows (f_n) do not converge uniformly on $\mathbb{R}$ to anything.]

We now look at several more examples illustrating these concepts.

Examples 1) Let $D = [0, 1]$ and $f_n(x) = x^n$. Then (f_n) converges pointwise, but NOT uniformly, to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Gap to fill in



Case by case analysis.

<u>For $x = 1$</u> : $f_n(x) = 1$ for all n

So $f_n(x) \to 1 = f(x)$ as $n \to \infty$.

[$f_n(1) \to 1 = f(1)$ as $n \to \infty$.]

For $0 \le x < 1$, we have $|x| < 1$.

So $f_n(x) = x^n \to 0 = f(x)$ as $n \to \infty$

Gap to fill in

So, for all x in $[0,1]$
we have $f_n(x) \to f(x)$ as $n \to \infty$.

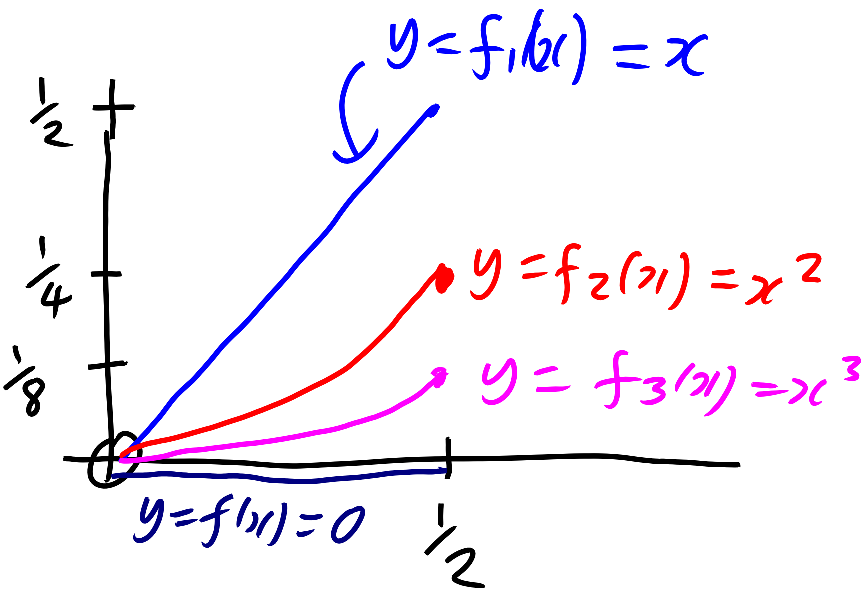
Exercise: (see previous years)
use function balls to show
convergence is not uniform here

OR Can quote following standard
theorem (see below): a uniform limit
of continuous functions must be cts.

Gap to fill in

2) Suppose $D = [0, 1/2]$ and again $f_n(x) = x^n$. Now (f_n) converges uniformly to the function which is identically 0 i.e. the function given by $f(x) = 0$ for all $x \in [0, 1/2]$ (we say it converges uniformly to 0).

Gap to fill in



Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough that $\frac{1}{2^N} \leq \varepsilon$.

Then for $n \in \mathbb{N}$ with $n \geq N$, for all $x \in [0, \frac{1}{2}]$, we have

$$|f_n(x) - f(x)| = |x^n - 0|$$

$$= x^n \leq \left(\tfrac{1}{2}\right)^n \leq \left(\tfrac{1}{2}\right)^N \leq \varepsilon.$$

[In terms of absorption, for all $n \geq N$ we have

Gap to fill in

$f_n \in \overline{B}_\varepsilon(f)$. so

$\overline{B}_\varepsilon(f)$ absorbs the sequence (f_n).]

This holds for each $\varepsilon > 0$

[N depends on ε]

so (f_n) converges to f uniformly on $[0, \frac{1}{2}]$.

Since (f_n) converges to 0 uniformly on $[0, \frac{1}{2}]$, (f_n) also converges to 0 pointwise on $[0, \frac{1}{2}]$.

N.b.

$$\begin{array}{c}\text{uniform convergence} \\ \Downarrow \quad \not\Uparrow \\ \text{pointwise convergence}\end{array}$$

3) Suppose $D = \mathbb{R}_+$ and $f_n : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} x/n & \text{if } 0 \le x \le n \\ 1 & \text{if } x > n. \end{cases}$$

Then the sequence of functions (f_n) converges to 0 pointwise, but not uniformly, on $\mathbb{R}_+$.

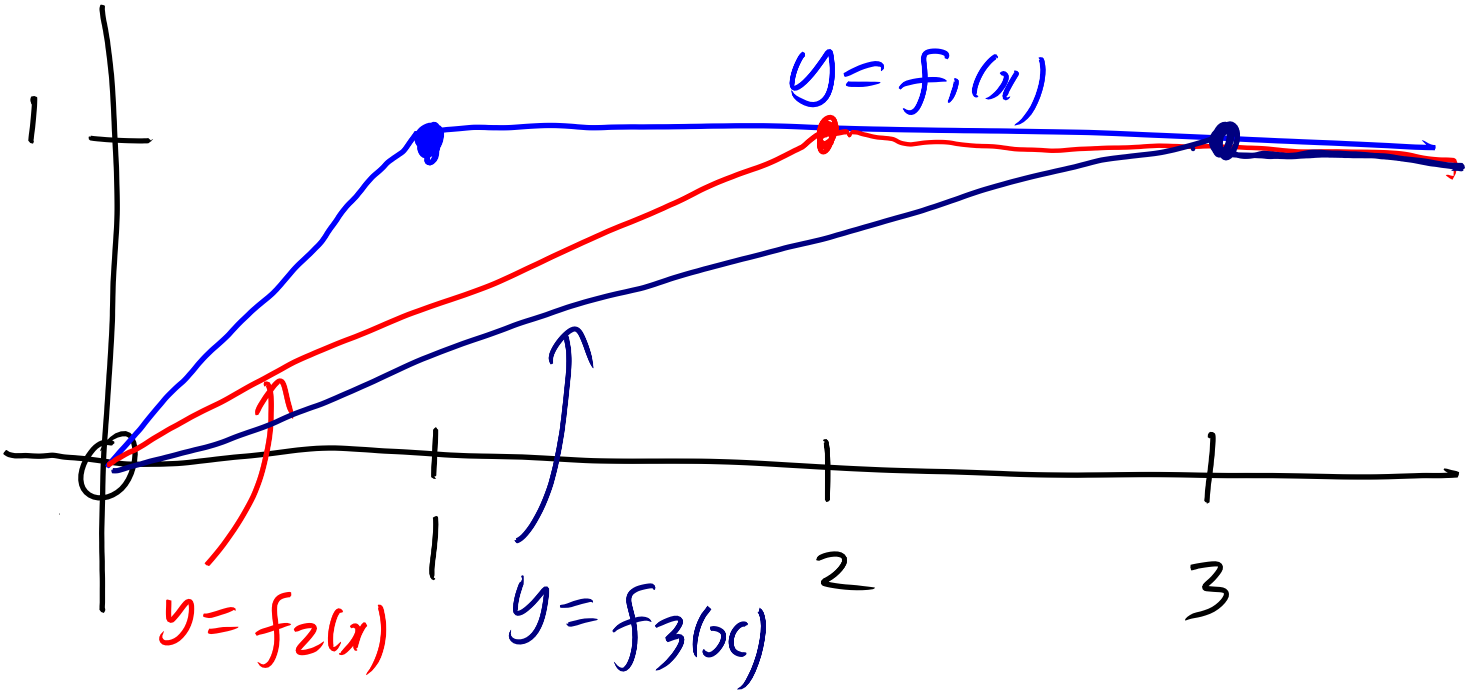
Gap to fill in

[Similar to earlier $f_n(x) = x/n$, but this time only defined on $\mathbb{R}^+$, and height is "capped" at 1.]

Note: $0 \le f_n(x) \le x/n$ for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$.

So can use sandwich theorem to see that $\lim_{n\to\infty} f_n(x) = 0$ for each x in $\mathbb{R}^+$. [Or use case by case analysis.]

Gap to fill in



$$f_n(n) = 1 \qquad \text{all } n$$

This prevents uniform convergence to 0 because $f_n \notin \overline{B_{\frac{1}{2}}(0)}$ (true all $n \in \mathbb{N}$).

constant function 0.

So $\overline{B_{\frac{1}{2}}(0)}$ does not absorb (f_n)

We will discuss a variety of methods to investigate the convergence of sequences of functions.

For an alternative approach, involving the **uniform norm**, see question sheets.

The above examples indicate that uniform convergence is much stronger than pointwise convergence and more difficult to establish.

One of the reasons that uniform convergence is important is that it has much better properties.

In the above examples, all of the functions f_n are continuous.

Unfortunately, as Example 1), shows the limit of a pointwise convergent sequence of continuous functions need not be continuous.

For uniformly convergent sequences the situation is much better.

The **proof** of the next theorem is **NEB** (not examinable as bookwork): see Wade's book, Theorem 7.9, if you are interested.

The **statement and applications of this theorem are examinable**.

Theorem 9.1.7 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \to \mathbb{R}$, and let (f_n) be a sequence of continuous functions from D to $\mathbb{R}$.

Suppose that the functions f_n converge uniformly on D to f.

Then f must also be continuous.

From now on **you may quote this theorem as standard**.

It may be summarized as follows:

Uniform limits of sequences of continuous functions are always continuous.

This theorem is one of the main reasons why uniform convergence is so important.

It sometimes gives you a quick way to see that certain pointwise convergent sequences do not converge uniformly.

If all the f_n are continuous but f isn't then your sequence cannot converge uniformly! (e.g. Example 1)

However, this trick does not always work.

Example 3) shows that a pointwise but non-uniform limit can sometimes be continuous.

We conclude this section with some additional standard facts which can sometimes help to establish that uniform convergence fails.

The proofs of these are an **exercise**.

Proposition 9.1.8 Let D be a non-empty subset of $\mathbb{R}^d$ and suppose that f is a **bounded** function from D to $\mathbb{R}$, i.e., $f(D)$ is a bounded subset of $\mathbb{R}$.

(i) Let $\varepsilon > 0$ and suppose that $g \in \bar{\boldsymbol{B}}_\varepsilon(f)$. Then g is also bounded on D.

(ii) Let (f_n) be a sequence of functions from D to $\mathbb{R}$. Suppose that all of the functions f_n are unbounded on D. Then (f_n) can not converge uniformly on D to f.

Another way to state this last result is:

> **It is impossible for a sequence of UNBOUNDED, real-valued functions from D to $\mathbb{R}$ to converge uniformly on D to a BOUNDED real-valued function.**

For example, functions $f_n(x) = x/n$ were unbounded, converged pointwise to bounded function 0.

So this results tells us this can not be uniform convergence

Since $g \in \bar{\boldsymbol{B}}_\varepsilon(f) \Leftrightarrow f \in \bar{\boldsymbol{B}}_\varepsilon(g)$, a similar proof shows the following:

> **It is impossible for a sequence of BOUNDED functions from D to $\mathbb{R}$ to converge uniformly on D to an UNBOUNDED real-valued function.**

Quiz

Which of these sequences of functions $\mathbb{R} \to \mathbb{R}$ converge to 0 (a) pointwise (b) uniformly?

(1) $f_n(x) = \sin(x/n)$

converges to 0 pointwise but not uniformly

(2) $f_n(x) = \frac{1}{n}\sin(nx)$ converges to 0 uniformly and hence pointwise.

(3) $f_n(x) = \cos(x/n)$

Does not tend to 0. $f_n(x) \to 1$ pointwise

(4) $f_n(x) = \frac{1}{n}\cos(nx)$.

but not uniformly

converges to 0 uniformly and hence also pointwise.