11 An introduction to Riemann Integration

The PROOFS of the standard lemmas and theorems concerning the Riemann Integral are NEB, and you will not be asked to reproduce proofs of these in full in the examination in January 2012.

However, you ARE expected to know the definitions and the statements of the results, and to know how to apply these results.

In particular, the EXAMPLES discussed in lectures ARE examinable.
11.1 Integration and antidifferentiation

From the modules G11CAL and G11ACF you will be familiar with integration as a form of antidifferentiation, and as ‘area under the curve’.

There are some problems here.

- Does it make sense to talk about ‘area’ for complicated regions in $\mathbb{R}^2$?
- Which functions have antiderivatives?

An antiderivative for a function $f$ is a differentiable function $F$ which satisfies $F' = f$.

Antiderivatives are also called primitives or indefinite integrals.
Recall that the **characteristic function** of a set $E$, $\chi_E$, is defined by $\chi_E(x) = 1$ if $x \in E$ while $\chi_E(x) = 0$ if $x \notin E$.

On Question Sheet 5 there is an example of a **function which has no antiderivative**: the characteristic function of a set with just one point.

**This shows that some care is needed!**

One of the main results of this chapter is the (first) **Fundamental Theorem of Calculus**, one implication of which is that **every continuous, real-valued function on an interval has an antiderivative**.

Of course, **some** discontinuous functions do have antiderivatives (can you think of an example?).
11.2 Partitions, areas and Riemann sums

Let $a$ and $b$ be real numbers with $a < b$.

For a **bounded**, real-valued function $f$ defined on $[a, b]$, we now discuss **partitions** $P$ of $[a, b]$ and the corresponding **Riemann upper sum** and **Riemann lower sum** for $f$ on $[a, b]$ (denoted by $U(P, f)$ and $L(P, f)$ respectively).

The bounded function $f$ and the interval $[a, b]$ will be fixed throughout the following definitions.

**Definition 11.2.1** A **partition** of $[a, b]$ is a **finite** set of points $P = \{x_0, x_1, \ldots, x_n\} \subseteq [a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$. The points $x_0, x_1, \ldots, x_n$ are called the **vertices** of $P$.

Gap to fill in
For $1 \leq k \leq n$, we set

$$M_k(P, f) = \sup\{f(t) \mid x_{k-1} \leq t \leq x_k\}$$

and

$$m_k(P, f) = \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\}.$$
The **Riemann upper sum for** $f$ **corresponding to** $P$, $U(P, f)$, and the **Riemann lower sum for** $f$ **corresponding to** $P$, $L(P, f)$, are defined by

\[
U(P, f) = \sum_{k=1}^{n} M_k(P, f)(x_k - x_{k-1})
\]

and

\[
L(P, f) = \sum_{k=1}^{n} m_k(P, f)(x_k - x_{k-1}).
\]
It is obvious that we always have $L(P, f) \leq U(P, f)$.

With a bit more work, we can prove the following fact (see books for details).

**Lemma 11.2.2** Let $P$ and $Q$ be partitions of $[a, b]$.

Then $L(P, f) \leq U(Q, f)$.

In words, this tells us the following.

The Riemann lower sum for $f$ corresponding to a partition $P$ of $[a, b]$ can not be greater than the Riemann upper sum for $f$ corresponding to a partition $Q$ of $[a, b]$, even if $P$ and $Q$ are different.
11.3 The Riemann integral

With $f$ and $[a, b]$ as above, the preceding lemma allows us to define the Riemann lower and upper integrals of $f$ over the interval $[a, b]$.

**Definition 11.3.1** The Riemann lower integral of $f$ over the interval $[a, b]$, $\int_{a}^{b} f(x) \, dx$ and the Riemann upper integral of $f$ over the interval $[a, b]$, $\int_{a}^{b} f(x) \, dx$ are defined by

\[
\int_{a}^{b} f(x) \, dx = \sup \{ L(P, f) : P \text{ is a partition of } [a, b] \} \\
\text{and} \\
\int_{a}^{b} f(x) \, dx = \inf \{ U(Q, f) : Q \text{ is a partition of } [a, b] \}.
\]
This in turn allows us to define Riemann integrability for $f$.

**Definition 11.3.2** The bounded function $f$ is **Riemann integrable** on $[a, b]$ if

$$\int_a^b f(x) \, dx = \int_a^b f(x) \, dx,$$

i.e., if the Riemann lower integral is equal to the Riemann upper integral.

In this case we define the Riemann integral of $f$ from $a$ to $b$ to be the common value:

$$\int_a^b f(x) \, dx = \int_a^b f(x) \, dx = \int_a^b f(x) \, dx.$$

**UNBOUNDED** functions on an interval $[a, b]$ are declared **NOT** to be Riemann integrable.

However they may have ‘improper’ integrals (as discussed in G11ACF).
Note the following facts for bounded, real-valued functions on an interval.

- The Riemann lower integral is always less than or equal to the Riemann upper integral.
- Every Riemann lower sum is less than or equal to the lower integral.
- Every Riemann upper sum is greater than or equal to the upper integral.

It is easy to show that constant functions are Riemann integrable, with the obvious integral (exercise).

We will see below that the family of Riemann-integrable functions is fairly large.

However, not all bounded functions are Riemann integrable.
**Example.** Let $f$ be the restriction of the characteristic function of the rationals, $\chi_\mathbb{Q}$, to $[0, 1]$.

So $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = 1$ for $x \in [0, 1] \cap \mathbb{Q}$, while $f(x) = 0$ for $x \in [0, 1] \cap \mathbb{Q}^c$.

This bounded function $f$ is **not** Riemann integrable on $[0, 1]$.

**Gap to fill in**
However, continuous functions are well-behaved.

**Theorem 11.3.3** Every continuous, real-valued function on an interval \([a, b]\) is Riemann integrable on \([a, b]\).

The converse to this theorem is **false**.

There are many discontinuous functions which are Riemann integrable.

For example (see Question Sheet 5), the characteristic function of a single-point set is discontinuous, but is nevertheless Riemann integrable.

**Gap to fill in**
The Riemann integral behaves as you expect a sensible notion of integration to behave.

**Theorem 11.3.4** Let \( f, g : [a, b] \to \mathbb{R} \) be Riemann integrable and \( \lambda \in \mathbb{R} \). Then \( f + g, \lambda f \) and \( |f| \) are also Riemann integrable and the following hold.

(a) \( \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \) (additivity).

(b) \( \int_a^b \lambda f(x) \, dx = \lambda \int_a^b f(x) \, dx. \)

(c) If \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]
(d) We always have

\[ \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx. \]

(e) For any \( c \in ]a, b[ \), we have that \( f \) is also Riemann integrable on \([a, c]\) and on \([c, b]\) and

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx. \]

We now come to the final (and main) result of this chapter.
Theorem 11.3.5 (Fundamental Theorem of Calculus, also known as the First Fundamental Theorem of Calculus)

Let $f$ be a continuous, real-valued function on $[a, b]$. For $x \in [a, b]$ define

$$F(x) = \int_a^x f(t) \, dt.$$ 

Then $F$ is continuous on $[a, b]$, and is differentiable on $]a, b[,$ with $F'(x) = f(x)$ for all $x \in ]a, b[.$

From this it follows easily that continuous, real-valued functions on intervals always have antiderivatives.

It also shows that antidifferentiation is the correct way to integrate continuous functions.

See Question Sheet 5 for more details.
As a concrete example, consider the function $G : ]0, \infty[ \to \mathbb{R}$ defined by

$$G(x) = \int_{0}^{x} \sin(-t^3) \, dt.$$ 

Then $G$ is differentiable on $]0, \infty[,$ and, for $x \in ]0, \infty[,$ we have

$$G'(x) = \sin(-x^3).$$

There is a more powerful integration theory due to Henri Lebesgue.

In this theory, $\chi_Q$ is integrable on $[0, 1]$ (with $\int_{0}^{1} \chi_Q(x) \, dx = 0$), and so are many other strange functions.

The Lebesgue integral is beyond the scope of this module, but it is an important tool in more advanced analysis and in probability theory.

**THE END**

Have a good holiday!