Extract from G11AN1 ANALYSIS lecture notes © Dr J.K. Langley 1993

(with slight modifications by Dr. Feinstein, 2000).

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6. L'Hopital's rule, Taylor's Theorem and Taylor Series

Indeterminate Forms

Consider the limit

$$\lim_{x \to 1} \frac{x^{16}+x-2}{x^2-1} .$$

Both numerator and denominator approach 0 as $x \to 1$. However, the limit may still exist. Such a limit is called an indeterminate form. To develop a quick way to evaluate similar limits, we first need:

Theorem 6.1 Cauchy's mean value theorem

Suppose that f,g are real-valued functions continuous on $[a, b]$ and differentiable on (a, b). Then there exists c in (a, b) such that

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c).$$

The **Proof** consists of just taking the function $h(x) = (f(b)-f(a))(g(x)-g(a))-(g(b)-g(a))(f(x)-f(a))$. Since $h(a) = h(b) = 0$ we obtain a c with $h'(c) = 0$ from Rolle's theorem.

Theorem 6.2 L'Hôpital's rule, first version

Let "lim" stand for any of $\lim_{x \to a+}$, $\lim_{x \to a-}$, $\lim_{x \to a}$, $\lim_{x \to +\infty}$, $\lim_{x \to -\infty}$. If $\lim f(x) = \lim g(x) = 0$ and

$$\lim \frac{f'(x)}{g'(x)} = L$$

(finite or infinite) exists, then

$$\lim \frac{f(x)}{g(x)}$$

exists and is the same.

Proof

We first consider the case of $\lim_{x \to a+}$. Since $\lim_{x \to a+} \frac{f'(x)}{g'(x)}$ exists, there must be some $\delta > 0$ such that $g'(s) \neq 0$ for $a < s \leq a+\delta$, since $f'(s)/g'(s)$ is defined. We set $f(a) = g(a) = 0$ and this makes f,g continuous on $[a, a+\delta]$. We also have $g(x) \neq 0$ for x in $(a, a+\delta]$, for otherwise Rolle's theorem would give us an s between a and x with $g'(s) = 0$.

Now take x such that $a < x < a+\delta$. Then by Theorem 6.1 there is a c_x in (a, x) such that

$$(f(x)-f(a))g'(c_x) = (g(x)-g(a))f'(c_x) .$$

This gives $f(x)/g(x) = f'(c_x)/g'(c_x)$. Now as $x \to a+$ we see that $c_x \to a+$ and so $f(x)/g(x) \to L$. The proof for $\lim_{x \to a-}$ is the same.

Now we consider the case where "lim" is $\lim_{x \to +\infty}$. Here we set $F(x) = f(1/x)$, $G(x) = g(1/x)$. Then $\lim_{x \to 0+} F(x) = \lim_{x \to 0+} G(x) = 0$. Now

$$L = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{f'(1/x)}{g'(1/x)} = \lim_{x \to 0+} \frac{(-1/x^2)f'(1/x)}{(-1/x^2)g'(1/x)} = \lim_{x \to 0+} \frac{F'(x)}{G'(x)} .$$

By the first part, the last limit is $\lim_{x \to 0+} F(x)/G(x) = \lim_{x \to +\infty} f(x)/g(x)$.

Examples 6.3

1. Consider $\lim_{x \to 1} \frac{x^{16}+x-2}{x^2-1}$. The rule applies and we can look at $\lim_{x \to 1} \frac{16x^{15}+1}{2x} = 17/2$. So the first limit is 17/2.

2. $\lim_{x \to 0} \frac{x-\sin x}{1-\cos x}$. Applying the rule, we look at $\lim_{x \to 0} \frac{1-\cos x}{\sin x}$. This is again indeterminate, but we can apply the rule again, and look at $\lim_{x \to 0} \frac{\sin x}{\cos x} = 0$. So the second limit is 0 and so is the first.

3. $\lim_{x \to +\infty} \frac{\log(1+e^x)}{x}$. This is slightly different, as numerator and denominator both tend to $+\infty$. We could convert it to the form covered above, but the computations will be very tricky. Instead, we wait for L'Hôpital's second rule (6.4).

4. $\lim_{x \to 1} \frac{x^2+1}{x^2-1}$. We CANNOT legitimately apply L'Hôpital's rule here, as the limit is NOT an indeterminate form. This is because $x^2+1 \to 2 \neq 0$ as $x \to 1$. In fact, the required limit does not exist.

5. $\lim_{x \to 0} \frac{x}{\sin x}$. Applying L'Hôpital's rule, we need to look at $1/(\cos x) \to 1$ as $x \to 0$. So the required limit is 1.

6. $\lim_{x \to 0} \frac{x^2\sin(1/x)}{\sin x}$. This is an indeterminate form. If we apply the rule, we need to look at $\lim_{x \to 0} \frac{2x\sin(1/x)-\cos(1/x)}{\cos x}$. However, this limit does not exist. This is because $x\sin(1/x)/\cos x \to 0$ as $x \to 0$, but $\cos(1/x)/\cos x$ has no limit as $x \to 0$.
This does not mean, however, that the required limit does not exist, as the rule says nothing about this case. In fact, since $\sin(1/x)$ is bounded, we see from Example 5 that the required limit is 0.

7. $\lim_{x \to +\infty} (1+1/x)^x$. If we take logarithms, we need to look at $\lim_{x \to +\infty} x\log(1+1/x) = \lim_{y \to 0+} \frac{\log(1+y)}{y}$. Applying the rule, we look at $\lim_{y \to 0+} \frac{1/(1+y)}{1} = 1$. So the required limit is e, since the exponential function is continuous at 1.

Now we prove the second version of L'Hôpital's rule.

Theorem 6.4

Let "lim" be as in 6.2. Suppose that $\lim f(x)$ is $+\infty$ or $-\infty$ and $\lim g(x)$ is $+\infty$ or $-\infty$.
If $\lim \frac{f'(x)}{g'(x)}$ exists and is L (finite or infinite) then $\lim \frac{f(x)}{g(x)}$ exists and is L.

Proof

We prove this only for $\lim\limits_{x \to a+}$. For $\lim\limits_{x \to +\infty}$ we can use the same trick as in 6.2.
Since $\lim f'(x)/g'(x)$ is assumed to exist, we see again that there must be some $\delta > 0$ such that $g'(s) \neq 0$ for $a < s \leqslant a+\delta$.
We prove simultaneously the cases $L \in \mathbb{R}$ and $L = +\infty$ (if $L = -\infty$ look at $-f/g$). Take an $\varepsilon > 0$ and an $M > 0$. We know that there is some $\rho > 0$ such that $\rho \leqslant \delta$ and such that $a < y < b = a+\rho$ implies that $f'(y)/g'(y)$ belongs to $(L-\varepsilon/2\ ,\ L+\varepsilon/2)$ (if L is finite) or $(2M\ ,\ +\infty)$ (if $L = +\infty$).
Now suppose that $a < x < b = a+\rho$. Then there exists a y with $x < y < b$ such that

$$(f(b)-f(x))g'(y) \;=\; (g(b)-g(x))f'(y)\ .$$

This gives

$$(f(b)-f(x))/(g(b)-g(x)) \;=\; f'(y)/g'(y)\ .$$

Dividing by $g(b)-g(x)$ is legitimate, for if $g(b) = g(x)$ we would obtain some s with $x < s < b$ and $g'(s) = 0$, which we have ruled out. Thus $(f(x)-f(b))/(g(x)-g(b))$ belongs to $(L-\varepsilon/2\ ,\ L+\varepsilon/2)$ or to $(2M\ ,\ +\infty)$. Now we write

$$\frac{f(x)}{g(x)} \;=\; \frac{f(x)}{f(x)-f(b)}\ \frac{f(x)-f(b)}{g(x)-g(b)}\ \frac{g(x)-g(b)}{g(x)}\ .$$

As $x \longrightarrow a+$, the first and last terms tend to 1, while the second term lies in $(L-\varepsilon/2\ ,\ L+\varepsilon/2)$ or $(2M\ ,\ +\infty)$. Therefore if x is close enough to a, then $f(x)/g(x)$ lies in $(L-\varepsilon\ ,\ L+\varepsilon)$ or $(M\ ,\ +\infty)$, which is exactly what we needed to show.

Examples 6.5

1. $\lim\limits_{x \to +\infty} \frac{\log(1+e^x)}{x}$. The rule 6.4 tells us to look at $\lim\limits_{x \to +\infty} \frac{e^x/(1+e^x)}{1} \;=\; 1$. So the required limit is e.

2. $\lim\limits_{x \to 0+} \frac{\log x}{\operatorname{cosec} x}$. We need to look at $\lim\limits_{x \to 0+} \frac{1/x}{-\cot x\ \operatorname{cosec} x} \;=\; \lim\limits_{x \to 0+} \frac{-\sin^2 x}{x \cos x}$. This is, using Example 5 of 6.3, equal to 0. So the required limit is 0.

3. $\lim\limits_{x \to +\infty} (\log x)^{1/x}$. We take logarithms and look at $\lim\limits_{x \to +\infty} \frac{\log\log x}{x}$. Applying 8.5 we look at $\lim\limits_{x \to +\infty} \frac{1/(x\log x)}{1} \;=\; 0$. The required limit is therefore $e^0 = 1$.

4. $\lim\limits_{x \to 0+} (\cos x)^{1/x^2}$. We take logarithms and look at $\lim\limits_{x \to 0+} \frac{\log(\cos x)}{x^2}$. The rule tells us to look at

$\lim_{x \to 0+} \frac{-\tan x}{2x} = \lim_{x \to 0+} \frac{-\sec^2 x}{2} = -1/2$. So the required limit is $e^{-1/2}$.

5. $\lim_{x \to 0+} (\sin x)^{1/\log x}$. We take logarithms and look at $\lim_{x \to 0+} \frac{\log(\sin x)}{\log x}$, and so at $\lim_{x \to 0+} \frac{\cot x}{1/x} = \lim_{x \to 0+} \frac{x \cos x}{\sin x} = 1$. The required limit is therefore e.

Theorem 6.6 Taylor's Theorem, or the *n*'th Mean Value Theorem

Suppose that n is a positive integer, and that f is a real-valued function which is n times differentiable on an interval containing the points a and $x \neq a$. Then there exists c lying strictly between a and x (i.e. in (a,x) or in (x,a)) such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^n}{n!} f^{(n)}(c).$$

Remarks

1. If we put $n = 1$ we get $f(x) = f(a) + (x-a)f'(c)$ which is the ordinary mean value theorem. 2. In general, c will depend on f, n and x. 3. There are other versions of this, with different forms for the remainder term, but this is probably the easiest to remember!

Proof of Taylor's theorem (THIS PROOF IS NON-EXAMINABLE, but you should make sure that you understand the theorem and can apply it)

We keep x fixed and for y lying between a and x we set

$$H(y) = \left(\sum_{k=0}^{n-1} \frac{(x-y)^k}{k!} f^{(k)}(y) \right) - f(x).$$

Then

$$H'(y) = -\sum_{k=1}^{n-1} \frac{(x-y)^{k-1}}{(k-1)!} f^{(k)}(y) + \sum_{k=0}^{n-1} \frac{(x-y)^k}{k!} f^{(k+1)}(y) = \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y).$$

Now we put

$$G(y) = H(y) - \frac{(x-y)^n}{(x-a)^n} H(a).$$

Then $G(a) = 0$ and $G(x) = H(x) = 0$. So by Rolle's theorem there exists a point c lying between a and x such that $G'(c) = 0$. This gives

$$H'(c) + n\frac{(x-c)^{n-1}}{(x-a)^n} H(a) = 0, \quad \text{and} \quad \frac{(x-c)^{n-1}}{(n-1)!} f^{(n)}(c) + n\frac{(x-c)^{n-1}}{(x-a)^n} H(a) = 0.$$

Therefore $H(a) = -\frac{(x-a)^n}{n!} f^{(n)}(c)$ which is what we need.

As an application we prove

Theorem 6.7 The Generalised Second Derivative Test

Suppose that $n \geqslant 2$ and that f is a real-valued function such that $f'(a) = ... = f^{(n-1)}(a) = 0$, and that $f^{(n)}(a)$ exists and is $\neq 0$.
If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.
If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.
If n is odd then f does not have a local maximum or a local minimum at a.

Proof

We suppose first that $f^{(n)}(a) > 0$ (if not, we can look at $-f$). Then

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} > 0.$$

So there is some $\delta > 0$ such that $f^{(n-1)}(s) > 0$ for all s in $(a,a+\delta)$ and $f^{(n-1)}(s) < 0$ for all s in $(a-\delta,a)$. Now suppose that $0 < |x-a| < \delta$. Then by Taylor's theorem (with n replaced by $n-1$) we have, for some s between a and x,

$$f(x) = f(a) + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(s).$$

If n is odd, this gives $f(x) > f(a)$ if $x > a$ and $f(x) < f(a)$ if $x < a$.
If n is even, we obtain $f(x) > f(a)$ for $x > a$ and for $x < a$. So we have a minimum.

Another application of Taylor's theorem is to estimation.

Example 6.8

Estimate $\cos(0.1)$ so that the error has absolute value less than 10^{-5}.
We use Taylor's theorem. With $f(x) = \cos x$ we have, for $n \in \mathbb{N}$,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{x^n}{n!} f^{(n)}(s)$$

for some s between 0 and x. Now $|f^{(n)}(s)|$ is certainly $\leqslant 1$. So we need to make $(0.1)^n/n! < 10^{-5}$, and $n = 4$ will do. Our estimate is $\sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}(0.1)^k = 199/200$.

The Taylor Series

Suppose that f is a real-valued function such that *all* the derivatives $f^{(n)}$ exist at a. Then we can form the Taylor series

$$T(x,a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Remarks

1. The special case where $a = 0$ is called the Maclaurin series of f. 2. $T(x,a)$ is, of course, a power series.
3. The obvious question to ask is: are $T(x,a)$ and $f(x)$ equal? Obviously $T(a,a) = f(a)$.

Example 1

Suppose that $f(x)$ is a polynomial. Let the degree of f be $n-1$. Then $f^{(n)}(x)$ is identically 0, and Taylor's theorem gives

$$f(x) \;=\; \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=\; \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=\; T(x,a)$$

so that the answer is always yes in this case.

Example 2

The functions sine and cosine equal their Maclaurin series for all x, i.e.

$$\sin x \,=\, x - x^3/3! + x^5/5! - \ldots.. \;\;,\;\; \cos x \,=\, 1 - x^2/4! + x^4/4! - \ldots.. \;\;.$$

You can prove this using Taylor’s theorem with $a = 0$, and f either sine or cosine. We get a remainder term $f^{(n)}(c)x^n/n!$, where c depends on x and n. However, $f^{(n)}(c)$ has absolute value at most 1, and so the remainder tends to 0 as $n \to \infty$.

However, in general, the answer to our question above is "not necessarily" as you will see on question sheet 5.