G12RAN Real Analysis

Exercises 1: Solutions to questions 6-10

There are many possible correct proofs of all these results and also many which are incorrect!

6 If either A or B is empty, then $A \times B$ is also empty, and so is countable.

Otherwise, there are sequences $(a_n) \subseteq A$, $(b_n) \subseteq B$ such that each element of A appears at least once in the sequence (a_n) [(a_n) "uses up" the elements of A] and each element of B appears at least once in the sequence (b_n).

But then each element of $A \times B$ appears at least once in the sequence

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), \dots .$$

(as in the proof that $\mathbb{N} \times \mathbb{N}$ is countable) and so $A \times B$ is countable.

NOTE: The sets A, B may be finite here.

Alternatively, you can quote the result that $\mathbb{N} \times \mathbb{N}$ is countable, and argue as follows: with $(a_n), (b_n)$ as above, define

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow A \times B$$

by

$$f\left((m, n)\right) = (a_m, b_n).$$

Then f is a surjection from $\mathbb{N} \times \mathbb{N}$ onto $A \times B$. Since $\mathbb{N} \times \mathbb{N}$ is countable the standard theory tells us that $A \times B$ is also countable.

7 Note: it is not enough to prove (by induction) that each of the finite unions $A_1 \cup A_2 \cup \cdots \cup A_n$ is countable. (After all, each of the sets $\{1, 2, \dots, n\}$ is *finite* but $\mathbb{N} = \{1, 2, 3, \dots\}$ is not a *finite* set, so why should the corresponding argument be valid using the word *countable* instead of *finite*? Some things can change when you look at the union of ALL of the sets and comparing with properties of the finite unions).

Strictly speaking, you should not assume that all of the sets A_n are infinite (though this is the main case of interest). Some may even be empty! To avoid any problems, you can set $B_n = A_n \cup \mathbb{N}$. Then, by Question 5, B_n is countable and infinite. Since $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} B_n$, it is enough to prove that $\bigcup_{n \in \mathbb{N}} B_n$ is countable (the result then follows, for every subset of a countable set is countable).

We know that $\mathbb{N} \times \mathbb{N}$ is countable. For each $n \in \mathbb{N}$ let f_n be a surjection (or a bijection if you like) from $\mathbb{N}$ onto B_n. Now define

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \bigcup_{n \in \mathbb{N}} B_n$$

by

$$f\left((i, j)\right) = f_i(j)$$

(for all (i,j) in $\mathbb{N}\times\mathbb{N}$).

Then, by the choice of the functions f_n, f is a surjection from $\mathbb{N}\times\mathbb{N}$ onto $\bigcup_{n\in\mathbb{N}} B_n$, and so $\bigcup_{n\in\mathbb{N}} B_n$ is countable, as claimed.

[As usual, there are many alternatives using sequences.]

8 Since $h(x)=0$ for *all* $x\neq 0$, (N.B. the value $h(0)$ is NOT relevant!) we have, clearly,

$$\lim_{x\to 0-} h(x) = \lim_{x\to 0+} h(x) = 0.$$

Thus $\lim_{x\to 0} h(x)$ exists and $=0$.

To find $\lim_{x\to 0} h\,(h(x))$, we should carefully establish what $h\,(h(x))$ is.

For $x\neq 0$, $h(x)=0$, so $h\,(h(x)) = h(0) = 1$.

For $x=0$, $h(x)=1$, so $h\,(h(x)) = h(1) = 0$.

Thus,

$$h\,(h(x)) = \begin{cases} 1 & \text{if } x\neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This gives us, clearly, $\lim_{x\to 0} h\,(h(x)) = 1$.

Note: this is *not* equal to $h\,(h(0))$, but (by chance) is equal to $h\,(\lim_{x\to 0}(h(x)))$ (this does not always happen either).

Note also that although $h(x)\to 0$ as $x\to 0$, $\lim_{x\to 0} h\,(h(x)) \neq \lim_{x\to 0} h(x)$. So the following argument is **FALSE**:

Set $y=h(x)$*. Then, as* $x\to 0$ *we also have* $y\to 0$*. Thus* $\lim_{x\to 0}(h(h(x)) = \lim_{y\to 0}(h(y)) = 0$.

The error here is that although y does indeed tend to zero (as x tends to zero), y does not tend to zero *through values unequal to zero*, and so the substitution is not valid.

9 This result is intuitively obvious, but it is good practice to write down a formal proof. Probably the easiest way to do this is using one of our standard versions of convergence for sequences: a sequence of real numbers (x_n) converges to the real number x if and only if, for all $\epsilon>0$, there are at most finitely many n with $x_n\notin(x-\epsilon,x+\epsilon)$. We prove separately that (a)→(b) and (b)⇒(a).

((a)⇒(b)) Given that (a) holds, we prove that b holds. Let $\epsilon>0$. We show that there are at most finitely many n with $c_n\notin(c-\epsilon,c+\epsilon)$. We already know that there are only finitely many k with a_k outside $(c-\epsilon,c+\epsilon)$, so (by the definition of c_n) at most finitely *odd* numbers n such that c_n is not in $(c-\epsilon,c+\epsilon)$. Similarly, there are only finitely many k with b_k outside $(c-\epsilon,c+\epsilon)$, and so there are at most finitely many *even* numbers n such that c_n is outside this open interval. Putting these together we see that there are at most finitely many n with c_n outside the open interval $(c-\epsilon,c+\epsilon)$, as required.

((b)⇒(a)) This time we are given that (c_n) converges to c. Let $\epsilon>0$. We know that there are at most finitely many n with $c_n\notin(c-\epsilon,c+\epsilon)$. Thus there are at most finitely many odd numbers n

with $c_n \notin (c - \epsilon, c + \epsilon)$, and so there are at most finitely many k with a_k not in $(c - \epsilon, c + \epsilon)$. It follows that (a_n) converges to c. The proof for (b_n) is the same, looking instead at the even numbers n where c_n is not in $(c - \epsilon, c + \epsilon)$.

10 Suppose, for contradiction, that $\lim_{x \to 0+} f(x)$ does not exist. Under the conditions of this question, this means that there must be two different sequences (x_n), (y_n) of positive real numbers both converging to 0 and such that the two sequences of images, $(f(x_n))$, $(f(y_n))$, converge to two different real numbers, say a and b. Now look at the sequence of positive real numbers $x_1, y_1, x_2, y_2, \ldots$. By question 9 this sequence also converges to 0. By the assumption in this question, the sequence $f(x_1), f(y_1), f(x_2), f(y_2), \ldots$ should converge to some real number. However, this is impossible because of question 9 again: our above choice gave us that $(f(x_n))$, $(f(y_n))$ converge to two different real numbers. This contradiction proves the result.