# G12RAN Real Analysis

## Exercises 2: Solutions to questions 1-5

1 (i) The definition from lectures tells us that we need to check the sequence of function values when the function is applied to an arbitrary sequence of the relevant type converging to the point in question. Here we are looking at $\lim_{x\to a+}$, so we must check sequences $(x_n)$ converging to $a$ and such that each $x_n > a$. Since the functions concerned are defined on $(a, b)$ we should insist that each $x_n$ is in $(a, b)$.

Let $(x_n) \subseteq (a, b)$ with $x_n \to a$ as $n \to \infty$. Then we know that $\lim_{n\to\infty} f(x_n) = L_1$ and $\lim_{n\to\infty} g(x_n) = L_2$ (directly from the definitions of the conditions on $f$ and $g$ given in the question). By the usual algebra of limits for real sequences, $\lim_{n\to\infty} (f(x_n) + g(x_n)) = L_1 + L_2$ and $\lim_{n\to\infty} (f(x_n)g(x_n)) = L_1L_2$. Since this holds for all such sequences $(x_n)$, the result follows.

(ii) Similarly, if $L_2 \neq 0$ and $g(x)$ is never 0 (for $x \in (a, b)$) then we can define the function $f/g$ as usual and note that, with $(x_n)$ as above, the usual algebra of limits gives us

$$\frac{f(x_n)}{g(x_n)} \to \frac{L_1}{L_2}$$

as $n \to \infty$, as required.

**NOTES**

The following fact is useful in some proofs (see below): given a real-valued function $f$, and two sequences $(x_n), (y_n)$ converging to a point $a \in \mathbb{R}$, then if both limits $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist and are *different*, then $\lim_{x\to a} f(x)$ does not exist.

If, in addition, all $x_n, y_n$ are $> a$ then we can deduce that $\lim_{x\to a+} f(x)$ does not exist. [A similar result holds for $\lim_{x\to a-} f(x)$.]

You do not need the full force of these assumptions: for example, if some sequence $(x_n)$ converging to $a$ with $x_n \neq a$ has the property that $\lim_{n\to\infty} f(x_n)$ does not exist, then $\lim_{x\to a} f(x)$ does not exist either. [Similar results hold for the other kinds of limit]

2 If $a$ is not an integer, then neither one-sided limit exists: say $a \in (m, m + 1)$, where $m \in \mathbb{Z}$. Then for all $x \in (m, m + 1)$, $[x] = m$. So, on this open interval,

$$f(x) = \begin{cases} m & \text{if } x \text{ is irrational,} \\ x & \text{if } x \text{ is rational.} \end{cases}$$

For any sequence of rational numbers $(x_n) \subseteq (m, m+1)$ converging to $a$, and with $x_n > a$, $f(x_n) \to a$ as $n \to \infty$. However, for any sequence $(y_n)$ of irrational numbers in this interval converging to $a$ and with $y_n > a$, $f(y_n) \to m \neq a$. This shows that $\lim_{x\to a+} f(x)$ does not exist, and a similar argument shows that $\lim_{x\to a-} f(x)$ does not exist either.

Now suppose that $a = m \in \mathbb{Z}$. For $x \in (m-1, m)$ we have

$$f(x) = \begin{cases} m-1 & \text{if } x \text{ is irrational,} \\ x & \text{if } x \text{ is rational.} \end{cases}$$

So, as above, $\lim_{x\to a-} f(x)$ does not exist. However, for $x \in (m, m+1)$,

$$f(x) = \begin{cases} m & \text{if } x \text{ is irrational,} \\ x & \text{if } x \text{ is rational.} \end{cases}$$

It is now easy to see that $f(x) \to m$ as $x \to m+$: for example you could note that, for $x \in (m, m+1)$, we have $m \le f(x) \le x$. Now take a sequence $(x_n)$ in $(m, m+1)$ converging to $m$ and apply the sandwich theorem to the inequality

$$m \le f(x_n) \le x_n.$$

So our answers are:

(a) The limit from the left does not exist anywhere.

(b) The limit from the right exists if and only if $a$ is an integer.

(c) In this particular case, when the limit from the right exists (i.e. when $a$ is an integer $m$) this limit is equal to $f(a)$ (which is also $m$).

3 This is similar to question 1.

Let $(x_n) \subseteq (a, b)$ with $x_n \to b$ as $n \to \infty$. We must show that

$$\lim_{n\to\infty} g(x_n) = L.$$

We know that

$$\lim_{x\to b-} f(x) = \lim_{x\to b-} h(x) = L,$$

so we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = L.$$

But $f(x_n) \le g(x_n) \le h(x_n)$ for all $n$, so the usual sandwich theorem for *sequences* of real numbers gives us $\lim_{n\to\infty} g(x_n) = L$ also, as required.

The result now follows.

4 The only point where the limits exist and are both equal to $f(a)$ is the point $a = \frac{1}{2}$.

Suppose $a \neq \frac{1}{2}$. Then, for any sequence of rational numbers $(x_n)$ converging to $a$ with $x_n > a$ you have

$$f(x_n) = x_n \to a \quad \text{as} \quad n \to \infty,$$

while, for any sequence of irrational numbers $(y_n)$ converging to $a$ with $y_n > a$ we have $f(y_n) = 1 - y_n \to 1 - a \neq a$. Thus $\lim_{x\to a+} f(x)$ does not exist. (You can use a similar argument to show that $\lim_{x\to a-} f(x)$ does not exist either, but this is unnecessary here.)

However, if $a = \frac{1}{2}$ then we see that for all $x$ in $\mathbb{R}$, $\left|(1-x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right|$, so, for any sequence $(x_n) \subseteq \mathbb{R} \setminus \{\frac{1}{2}\}$ with $x_n \to \frac{1}{2}$ as $n \to \infty$, we have

$$\left|f(x_n) - f(\tfrac{1}{2})\right| = \left|f(x_n) - \tfrac{1}{2}\right| = \left|x_n - \tfrac{1}{2}\right| \to 0 \quad \text{as} \quad n \to \infty,$$

so $f(x_n) \to f(\frac{1}{2})$ as $n \to \infty$.

5 No, there is no such function. The trick is to prove it! Here are two methods.

**Method I: sup and inf.** *Suppose, for contradiction, that such a function $f$ exists.* Set

$$E_1 = \{x \in \mathbb{R} : f(x) < 0\}$$

and set

$$E_2 = \{x \in \mathbb{R} : f(x) > 0\}.$$

By assumption, $\mathbb{R} = E_1 \cup E_2$. Clearly $E_1 \cap E_2 = \emptyset$. Also, since $f$ is non-decreasing we have, for all $x$ in $E_1$ and $y$ in $E_2$, that $x < y$. Again, by our assumptions on $f$, $E_1$ and $E_2$ are non-empty.

Thus $E_1$ is non-empty and bounded above: in fact every element of $E_2$ is an upper bound for $E_1$. It follows that $\sup(E_1)$ exists and is a lower bound for $E_2$. Since $E_2$ is non-empty, we have that $\sup(E_1) \leq \inf(E_2)$.

Set $a = \sup(E_1)$, $b = \inf(E_2)$. We have that $a \leq b$. Since $E_1 \subseteq (-\infty, a]$ and $E_2 \subseteq [b, \infty)$, the fact that $\mathbb{R} = E_1 \cup E_2$ forces $a = b$.

Now consider two cases:

(a) $a \in E_1$. Then, since $f$ is nondecreasing, $f(x) \leq f(a) < 0$ for all $x$ in $E_1$, while $f(x) > 0$ for all $x$ in $E_2$, so $f(x) \neq \frac{1}{2}f(a)$ for all $x$ in $\mathbb{R}$, contradicting the fact that $f$ is onto.

(b) $a \in E_2$. But then $0 < f(a) \leq f(x)$ for all $x$ in $E_2$ (N.B. $a = \inf(E_2)$), and $f(x) < 0$ for all $x$ in $E_1$, so $f(x) \neq \frac{1}{2}f(a)$ for all $x$ in $\mathbb{R}$, again contradicting the fact that $f$ is onto.

*Both cases lead to a contradiction*, so no such $f$ can exist.

**Method II: nested intervals theorem.** This method is probably the slickest! Again, *suppose for contradiction* that such a function $f$ *does* exist. Then, for each $n \in \mathbb{N}$ we can find real numbers $a_n$, $b_n$ such that $f(a_n) = -1/n$ and $f(b_n) = 1/n$. Since $f$ is nondecreasing, we see immediately that $a_n < b_n$, $a_n < a_{n+1}$ and $b_n > b_{n+1}$. Thus the intervals $[a_n, b_n]$ are nested, and the nested intervals theorem tells us that there exists (at least one) $c$ in $\cap_{n\in\mathbb{N}}[a_n, b_n]$. (There may be infinitely many such $c$.) For any such point $c$, we have (for all $n \in \mathbb{N}$) $f(a_n) \leq f(c) \leq f(b_n)$, i.e. $-1/n \leq f(c) \leq 1/n$. Since this holds for all $n \in \mathbb{N}$ we must have $f(c) = 0$, and this *contradicts* our assumption on $f$. Thus no such function $f$ can exist.