G12RAN Real Analysis

Exercises 2: Solutions to questions 6-12

6 Let f be the function defined in question 2. Since, for all $a \in \mathbb{R}$, the limit from the left $\lim_{x\to a-} f(x)$ does not exist, there are no points of $\mathbb{R}$ at which this function is continuous. [For f to be continuous at a both one-sided limits must exist and must be equal to $f(a)$.]

7 The only possible problems are at the points $x = -1$ and $x = 1$. This is because it is standard that polynomials are continuous, and on each one of the open intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ the function is equal to the same polynomial throughout the open interval. We must make the values at the points -1 and 1 match up correctly. Since $f(-1) = f(1) = 2$, we are led to the equations

$$\begin{aligned} a \times (-1) + b &= 2 \\ \text{and} \quad -a \times (1) + 2b &= 2 \end{aligned}$$

giving $b = 0$, $a = -2$. The function is thus

$$f(x) = \begin{cases} -2x & \text{if } x < -1, \\ x^2 + 1 & \text{if } -1 \le x \le 1, \\ 2x & \text{if } x > 1. \end{cases}$$

Note (again using the continuity of polynomials) that we really do have $\lim_{x\to -1-} f(x) = 2$ and $\lim_{x\to 1+} f(x) = 2$.

Easy exercise: Sketch this function!

8 There are many examples. The easiest is probably

$$f(x) = \begin{cases} x & (x \in \mathbb{Q}), \\ x + 1 & (x \in \mathbb{R}\backslash\mathbb{Q}). \end{cases}$$

Certainly $f : \mathbb{R} \to \mathbb{R}$. To see that f is surjective, let $y \in \mathbb{R}$. Then

Case (i) $y \in \mathbb{Q}$: then $f(y) = y$,
Case (ii) $y \in \mathbb{R}\backslash\mathbb{Q}$: then $f(y - 1) = y$.

Thus every $y \in \mathbb{R}$ is in the image of f, i.e. f is surjective from $\mathbb{R}$ onto $\mathbb{R}$.

To see that f is discontinuous at every point of $\mathbb{R}$, we use the same method as in questions 2 and 4.

Let $a \in \mathbb{R}$. Take a sequence $(x_n) \subseteq \mathbb{Q}$ with $\lim_{n\to\infty} x_n = a$, and a sequence $(y_n) \subseteq \mathbb{R}\backslash\mathbb{Q}$ with $\lim_{n\to\infty} y_n = a$. Then $f(x_n) = x_n \to a$ as $n \to \infty$ but $f(y_n) = y_n - 1 \to a - 1$ as $n \to \infty$. Since $(x_n), (y_n)$ both converge to a, the above shows that f is discontinuous at a ($f(a)$ cannot equal both a and $a - 1$).

Thus f is discontinuous at every point of $\mathbb{R}$, as required.

9 Say $p(x) = a_0 + a_1x + \cdots + a_nx^n$ where n is odd and $a_n \neq 0$. Dividing by a_n does not change the result, so we may assume that $p(x)$ has the form $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$. When $|x|$ is large, $p(x)$ has the same sign as x (can you prove this?) so choose $a < 0$ with $p(a) < 0$ and $b > 0$ with $p(b) > 0$. Then, by the Intermediate Value Theorem (IVT), there exists $c \in [a, b]$ with $p(c) = 0$.

10 To see that f is discontinuous at rationals is easy. Let $x = p/q$ where p, q are positive integers with no common factor (and with $p < q$ so that $x \in (0, 1)$). Then $f(x) = 1/q > 0$.

Let (x_n) be a sequence of *irrational* numbers in $(0, 1)$ with $x_n \to x$ as $n \to \infty$. Then $f(x_n) = 0$ for all n, so $f(x_n) \not\to f(x)$ as $n \to \infty$. This proves that f is discontinuous at x.

Trickier is to see that f is continuous at irrational x. Here I think that $\varepsilon - \delta$ is the best way: here is an example of such a proof. (We will look at $\varepsilon - \delta$ methods further in Chapter 7.)

Given $x \in (\mathbb{R}\backslash\mathbb{Q}) \cap (0, 1)$, and given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $1/N < \varepsilon$.

Then set

$$E = \left\{ \frac{p}{q} : p, q \in \mathbb{N},\ 1 \leq q \leq N,\ 1 \leq p \leq q \right\} \cup \{0\}.$$

E is a *finite* set, so we can set $\delta = \min\{|x - y| : y \in E\}$.

Claim: For all y in $(x - \delta, x + \delta)$ we have $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. To see this, let $y \in (x - \delta, x + \delta)$.

Case (i): $y \in \mathbb{R}\backslash\mathbb{Q}$. Then $f(y) = 0$, so

$$|f(y) - f(x)| = 0 < \varepsilon.$$

Case (ii): $y \in \mathbb{Q}$. Then $y = p/q$ for some $p, q \in \mathbb{N}$ with p, q having no common factors, and $1 \leq p < q$. But since $|y - x| < \delta$, y cannot be in E, and so q must be $> N$. Thus

$$|f(y) - f(x)| = |f(y)| = \frac{1}{q} < \frac{1}{N} < \varepsilon.$$

In both cases, we have $|f(y) - f(x)| < \varepsilon$, so this holds for all y in $(x - \delta, x + \delta)$. Thus f is continuous at x, as claimed.

11 Set $g(x) = f(x) - x$. Then $f(x) = x \iff g(x) = 0$. But g is continuous $[0, 1] \to \mathbb{R}$, $g(0) \geq 0$ and $g(1) \leq 0$. (N.B. $f : [0, 1] \to [0, 1]$.) Thus, by the intermediate value theorem there must be an $x \in [0, 1]$ with $g(x) = 0$. For such x we have $f(x) = x$, as required.

[Points where $f(x) = x$ are called "fixpoints" or "fixed points" for f. This question shows that every continuous map from $[0, 1]$ to itself has at least one fixed point.]

12 There are many ways to prove this. One is to prove the result first for closed intervals, and then deduce the result for open intervals. Others involve careful case-by-case analysis of several cases.

We are given: $f : (a, b) \to \mathbb{R}$, f is continuous, and f is injective. So for all x, y in (a, b) with $x \neq y$ we have $f(x) \neq f(y)$. [Thus $f(x) < f(y)$ or $f(x) > f(y)$. This will be used frequently below.] We are asked to prove that f is monotone. Now f is *not* monotone if and only if there are points $c_1, c_2, d_1, d_2 \in (a, b)$ such that

$$c_1 < c_2, d_1 < d_2, f(c_1) < f(c_2), \text{ and } f(d_1) > f(d_2). \quad (*)$$

(However, we do not know whether or not $c_i < d_j$, $1 \leq i, j \leq 2$). We are required to show that $(*)$ never happens. Case by case analysis using the intermediate value theorem shows that no such c_1, c_2, d_1, d_2 can exist, but there are a lot of cases! Perhaps better is to prove successively:

(A) f is strictly monotone on every subset of (a, b) consisting of 3 points;

(B) f is strictly monotone on every finite subset of (a, b);

(C) f is (strictly) monotone on (a, b).

N.B. For $E \subseteq (a, b)$, f is strictly monotone on E if f is strictly increasing on E or f is strictly decreasing on E.

To prove (A). (A) says that for $a < x_1 < x_2 < x_3 < b$ we must have either $f(x_1) < f(x_2) < f(x_3)$ or $f(x_1) > f(x_2) > f(x_3)$, or in other words $f(x_2) - f(x_1)$ and $f(x_3) - f(x_2)$ have the same sign, $+$ or $-$. *Suppose this is false.* Then we can find $a < x_1 < x_2 < x_3 < b$ with $f(x_2) - f(x_1)$ and $f(x_3) - f(x_2)$ having opposite signs. By symmetry we may assume that $f(x_1) < f(x_2) > f(x_3)$.

[**Exercise:** draw a sketch to illustrate this situation.]

Set

$$y = \frac{1}{2}(\max\{f(x_1), f(x_3),\} + f(x_2))$$

so that

$f(x_1) < y < f(x_2)$ and $f(x_3) < y < f(x_2)$. By the intermediate value theorem there must be $c_1 \in (x_1, x_2)$ with $f(c_1) = y$ and also $c_2 \in (x_2, x_3)$ with $f(c_2) = y$. But this contradicts the fact that f is injective on (a, b). This contradiction proves (A).

(B) Now suppose that $a < x_1 < x_2 < \ldots < x_n < b$ with $n \geq 3$. By (A) we know that for $1 \leq i \leq n-2$, $f(x_{i+1}) - f(x_i)$ has the same sign as $f(x_{i+2}) - f(x_i)$. So *all* of these must have the same sign, and f is strictly monotone on $\{x_1, \ldots, x_n\}$. The cases where $n \leq 2$ are trivial.

(C) It now follows (from the cases $n \leq 4$ of (B)) that no c_1, c_2, d_1, d_2 can be found satisfying $(*)$ above. (C) follows.