G12RAN Real Analysis

Exercises 5: Solutions to questions 1-4

1 With $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f^{(3)}(x) = -\cos(x)$.

We take the first 3 terms of the Taylor series for f about the point 0 (i.e. the Maclaurin series for f). Taylor's theorem gives

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(c)}{6}x^3$$

for some c between 0 and x.

But $f(0) = 0$, $f'(0) = 1$, $f^{(2)}(0) = 0$ and $f^{(3)}(c) = -\cos(c)$, so

$$f(x) = x - \frac{\cos(c)}{6}x^3$$

which gives

$$\begin{aligned} |f(x) - x| &= \left|\tfrac{\cos(c)}{6}x^3\right| \\ &\leq \tfrac{|x^3|}{6} \end{aligned}$$

as required.

[You could treat $x = 0$ as a special case, or accept the above argument with $0 = c = x$.]

2 (i) Here $f(x) = \sqrt{x} = x^{1/2}$ so $f'(x) = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. [Remember that you can justify this by noting that $x^a = \exp(a \log x)$, as on question sheet 4. Do you know a *direct* way to prove that the derivative of $\sqrt{x}$ is as claimed?]

(ii) Since $f'(x)$ diverges to $+\infty$ as $x \to 0+$, the derivative is unbounded and so, by a standard result in the notes, f is not Lipschitz continuous.

(iii) This follows immediately from squaring both sides. (Remember that we always take the non-negative square root in this module!)

(iv) When $0 < y \leq x$, set $a = y$, $b = x - y$ and then (iii) gives

$$\sqrt{x} = \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} = \sqrt{y} + \sqrt{x-y}.$$

Clearly $\sqrt{x} \geq \sqrt{y}$, so we have, in this case, $0 \leq f(x) - f(y) \leq \sqrt{x-y}$. Obviously, if $0 < x \leq y$ we obtain similarly $0 \leq f(y) - f(x) \leq \sqrt{y-x}$. Thus, in all cases, we have

$$|f(x) - f(y)| \leq \sqrt{|x-y|}.$$

Now suppose that (x_n), (y_n) are sequences in $(0, \infty)$ with $\lim_{n\to\infty} |x_n - y_n| = 0$. Then we also have $\lim_{n\to\infty} \sqrt{|x_n - y_n|} = 0$, and we know that

$$0 \leq |f(x_n) - f(y_n)| \leq \sqrt{|x_n - y_n|}.$$

By the sandwich theorem we must have $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$. Thus f is uniformly continuous on $(0, \infty)$, as claimed.

The above used the definition of uniform continuity in terms of sequences. An alternative approach, using ε and δ, is to note that given $\varepsilon > 0$ you can take $\delta = \delta(\varepsilon) = \varepsilon^2$. Then the above calculations show that if x and y are positive real numbers with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$, as required for uniform continuity. (Note that δ depends on ε but does not depend on x and y.)

3 This time $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f^{(2)}(x) = -\cos(x)$, $f^{(3)}(x) = \sin(x)$, $f^4(x) = \cos(x)$, etc.

Since $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, and $a_n = \frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}$, we see that

$$a_0 = \frac{1}{\sqrt{2}}, \quad a_1 = \frac{-1}{\sqrt{2}}, \quad a_2 = \frac{-1}{2!\sqrt{2}}, \quad a_3 = \frac{1}{3!\sqrt{2}}, \quad a_4 = \frac{1}{4!\sqrt{2}} \quad \text{etc}$$

giving

$$T\left(x, \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \dots .$$

[This series does, in fact, converge to $\cos(x)$ for all x.]

4 To see how this works, let us start by checking the first derivative.

For $x \neq 0$ there is no problem differentiating f by the chain rule, to obtain

$$f'(x) = \left(\frac{2}{x^3}\right) \exp\left(-\frac{1}{x^2}\right).$$

So take $p_1(t) = 2t^3$, and then $f'(x) = p_1\left(\frac{1}{x}\right) \mathrm{e}^{-\frac{1}{x^2}}$ for $x \neq 0$. However we need to check that $f'(0)$ exists and is 0.

For $x \neq 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{\exp\left(-\frac{1}{x^2}\right)}{x}.$$

Now $\exp\left(\frac{1}{x^2}\right) > \frac{1}{x^2}$ (since $\exp(y) = 1 + y + \frac{y^2}{2!} + \cdots$) so

$$\left|\frac{1}{x} \exp\left(-\frac{1}{x^2}\right)\right| \leq \frac{1}{|x|} \times \frac{1}{\left(\frac{1}{x^2}\right)} = |x|.$$

Hence $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ by the sandwich theorem and $f'(0) = 0$, as claimed, so the result of the question is true when $n = 1$.

A slight variation in the above argument shows that, for all $k \in \mathbb{N}$,

$$\lim_{x\to 0} \frac{\exp\left(-\frac{1}{x^2}\right)}{x^k} = 0,$$

and so, for any polynomial $p(t)$

$$\lim_{x\to 0} p\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) = 0.$$

Now suppose that $n > 1$ and the result of the question is true for $n-1$, so that $f^{(n-1)}(0) = 0$, while $f^{(n-1)}(x) = p_{n-1}\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right)$ for $x \neq 0$, where p_{n-1} is a polynomial.

Then, for $x \neq 0$, we can differentiate $f^{(n-1)}$ by the chain rule, and

$$(f^{(n-1)})'(x) = \left[\left(\frac{2}{x^3}\right)p_{n-1}\left(\frac{1}{x}\right) - \left(\frac{1}{x^2}\right)p'_{n-1}\left(\frac{1}{x}\right)\right]\exp\left(-\frac{1}{x^2}\right)$$

so that $f^{(n)}(x) = p_n\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right)$, where $p_n(t) = 2t^3p_{n-1}(t) - t^2p'_{n-1}(t)$, which is a polynomial because p_{n-1} is.

It remains to check that $f^{(n)}(0)$ exists and is 0. But this is true because, by our remarks above,

$$\lim_{x\to 0}\left(\frac{1}{x}\right)p_{n-1}\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) = 0$$

and so

$$\lim_{x\to 0}\left(\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right) = 0,$$

as required.

The induction may now proceed, and the result holds for all $n \in \mathbb{N}$.