## G12RAN: Real Analysis

## Exercises for the enthusiast

Please note that Dr Feinstein did not invent these prize problems (they are well-known). For each of the prize problems, a prize will be awarded by Dr Feinstein for the first correct solution (including justification sufficient to convince Dr Feinstein!) handed in (by the deadline below) by a student registered for at least one of the modules G11LMA: Linear Mathematics or G12RAN: Real Analysis in the academic year 2002-3. Solutions should be handed directly to Dr Feinstein on or before May 9 2003. Note that the solutions to these prize problems need not necessarily have anything to do with G11LMA Linear Mathematics or G12RAN Real Analysis.

**Prize problem 1.** *[Prize: 10 pounds]* Give (with proof, of course!) an explicit example of a positive real number $t$ such that the sequence $\cos(n!t)$ does not converge.

**Prize problem 2.** *[Prize: 10 pounds]* Suppose that a rectangle is formed from finitely many smaller rectangles, and that each of the smaller rectangles has the property that at least one of its sides has integer length. Show that the original rectangle must also have at least one side of integer length.

**Prize problem 3.** *[Prize: 10 pounds]* Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$ which is infinitely differentiable, i.e. $f$ can be differentiated as many times as you like at all points of $\mathbb{R}$. Denote the $n$th derivative of $f$ by $f^{(n)}$ (by convention, $f^{(0)}$ just means $f$).

Suppose that for all $a \in \mathbb{R}$ there exists at least one integer (possibly depending on the point $a$) $n \in \{0, 1, 2, \ldots\}$ such that $f^{(n)}(a) = 0$ (i.e. at every point of $\mathbb{R}$ there is a derivative of some order of $f$ which is 0 at that point). Prove that $f$ must be a polynomial function.

**Prize problem 4.** *[Prize: 10 pounds]* Does there exist an uncountable collection of subsets of $\mathbb{N}$ with the property that, for each pair of sets $A$, $B$ in the collection, either $A$ is a subset of $B$ or $B$ is a subset of $A$?

**Prize problem 5.** *[Prize: 15 pounds]* Let $f$ be a real-valued function defined on the unit square $[0,1]^2$. Suppose that, for each fixed $x \in [0,1]$, $f(x,y)$ is a polynomial in $y$, and that, for each fixed $y \in [0,1]$, the function $f(x,y)$ is a polynomial in $x$. Prove that $f$ must be a polynomial in the two variables $x$ and $y$.

**Prize problem 6.** *[Prize: 7 pounds]* Show that there DOES exist a function $f$ from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$ such that $f(x,y)$ is NOT a polynomial in $x$ and $y$, and yet for each fixed $x \in \mathbb{Q}$, $f(x,y)$ is a polynomial in $y$, and, for each fixed $y \in \mathbb{Q}$, the function $f(x,y)$ is a polynomial in $x$.