Density of the rationals and irrationals in terms of sequences

The following result reflects the fact that the rationals and the irrationals are both dense in $\mathbb{R}$. This result is standard, and you should know the statement and be able to apply this result (as in the second problems class). You may always assume it without proof as long as you state the part that you need clearly.

The proof of this result is non-examinable

Theorem. Let a, b be real numbers with $a < b$. Then there are sequences of rational numbers (x_n), (w_n) in (a, b) such that (x_n) converges to a and (w_n) converges to b. Similarly there are sequences of irrational numbers (y_n), (z_n) in (a, b) such that (y_n) converges to a and (z_n) converges to b. Moreover it is possible to find such sequences such that (x_n) and (y_n) are strictly decreasing sequences and (w_n), (z_n) are strictly increasing sequences.

Proof (non-examinable). The four proofs needed here are all essentially identical, and use the fact that whenever c, d are real numbers with $c < d$ then there is at least one rational number in (c, d) and there is at least one irrational number in (c, d) (of course there are really infinitely many of each). We give one of many possible proofs of the existence of a sequence (x_n) with the properties above. Modifying this proof to give you proofs for the other three sequences is an easy exercise for you!

We want to find a sequence (x_n) of rational numbers in (a, b) with the property that $x_1 > x_2 > x_3 > \cdots$ and such that $x_n \to a$ as $n \to \infty$. We begin, instead, by choosing ANY sequence of numbers (b_n) in (a, b) such that $b_1 > b_2 > b_3 > \cdots$ and such that $b_n \to a$ as $n \to \infty$. For example, you could set $b_n = a + \dfrac{b-a}{2^n}$. We have no idea at this point whether the numbers b_n are rational or irrational. However, since we always have $b_n > b_{n+1}$, we know that we can find at least one rational number in (b_{n+1}, b_n). Choose such a rational number and call it x_n. This gives us a sequence of rational numbers x_n and we have

$$b > b_1 > x_1 > b_2 > x_2 > b_3 > \cdots > a$$

so certainly $(x_n) \subseteq (a, b)$ and the sequence (x_n) is strictly decreasing. Finally, since

$$b_{n+1} < x_n < b_n$$

and

$$\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} b_n = a,$$

we may apply the sandwich theorem to deduce that $\lim_{n\to\infty} x_n = a$ also, as required.