G12RAN: Real Analysis
1. Properties of the real numbers

II. Further properties

The nested intervals theorem

The following theorem, which we shall deduce from the Monotone Sequence Theorem, will be used many times in this module.

Theorem 1.6 (Nested intervals theorem) Let $(a_n), (b_n)$ be sequences of real numbers with $a_n \leq b_n$ for all n. Suppose further that $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$. Then there is at least one real number c with the property that c belongs to *all* of the intervals $[a_n, b_n]$. ■

The point c need not be unique. In fact, any point between $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ (inclusive) will do. However, if we know that the lengths of the intervals tend to zero (i.e. $\lim_{n\to\infty}(b_n - a_n) = 0$) then the point c *is* unique.

Intersections and unions of infinitely many sets

Suppose that we have a sequence of sets, $A_1, A_2, A_3, \ldots$. It is possible to form the following intersections and unions:

$$\bigcap_{n\in\mathbb{N}} A_n = \{x : x \text{ is in every one of the sets } A_n\};$$

$$\bigcup_{n\in\mathbb{N}} A_n = \{x : x \text{ is in at least one of the sets } A_n\}.$$

These may also be denoted, respectively, by

$$\bigcap_{n=1}^{\infty} A_n$$

and

$$\bigcup_{n=1}^{\infty} A_n.$$

However, **please note** that this notation does *not* mean that there is a set called A_∞! It just means take the intersection/union of all of the sets A_n, where $n \in \mathbb{N}$.

In terms of intersections, the conclusion of the nested intervals theorem is that, *under the conditions of that theorem,*

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset :$$

in fact, set $a = \lim_{n\to\infty} a_n$ and set $b = \lim_{n\to\infty} b_n$. Then

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] = [a, b].$$

If $\lim_{n\to\infty}(b_n - a_n) = 0$ then $a = b$ and so the closed interval $[a, b]$ has exactly one point in it.

The nested intervals theorem fails if you use open intervals (a_n, b_n) instead of closed intervals $[a_n, b_n]$ (see question sheet 1). It also relies on the completeness of $\mathbb{R}$. The statement with closed intervals would become false if you restricted attention to the rational numbers, in the sense shown by the following exercise.

Exercise Let α be an irrational number (if you like you may assume that $\alpha = \sqrt{2}$). Show that there are sequences of rational numbers (a_n), (b_n), both converging to α and such that (a_n) is a strictly increasing sequence, while (b_n) is a strictly decreasing sequence. Show further that there is no rational number c in $\bigcap_{n\in\mathbb{N}}[a_n, b_n]$.