G12RAN Real Analysis: Exercises 4

You should attempt questions 1 to 3 in the fourth problem class. If you finish these questions early you can look at questions 4 and 5. **You should make sure that you are familiar with all the concepts mentioned in questions 1-5 before the problem class.**

Answers to questions 7 and 8 should be handed in at the end of the lecture on Friday 29th November.

Note that quality of exposition is very important: poor exposition/expression in the examination will cost marks. You must always justify your answers.

1. Set $f(x) = x^3 + x - \cos x$. Show that for all $x \in \mathbb{R}$ we have $f'(x) > 0$. What does this tell you about the function f?

2. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable at 0. Suppose that $f(1/n) = 0$ for all $n \in \mathbb{N}$.

(i) Prove that $f(0) = 0$. (This fact does not requires the differentiability of f.)

(ii) Prove that $f'(0) = 0$. (This uses all the assumptions above along with the conclusion of part (i).)

For the next question, **you may assume** that $\arcsin(x)$ is continuous on $[-1, 1]$, and differentiable on $(-1, 1)$ with derivative $1/\sqrt{1-x^2}$ there. (The function $\arcsin(x)$ is also denoted by $\sin^{-1}(x)$).

3. Find the greatest and least values taken by the function $x \arcsin(x) + \sqrt{1-x^2}$ in the range $-1 \le x \le 1$. [You should check endpoints and any stationary points in the relevant range.]

4. Does there exist a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f'(0) = 0$ and yet $f'(x) > 1$ for all $x > 0$?

Hint: Suppose that f is such a function. For $x > 0$, apply the mean value theorem with $a = 0$ and $b = x$ to show that

$$\frac{f(x) - f(0)}{x - 0} > 1.$$

(NB you are not allowed to assume that f' is continuous.)

5. Using the mean value theorem or otherwise, show that the function $f(x) = \cos(\log(x))$ is Lipschitz continuous on $(1, \infty)$.

6. Let f be a differentiable, real-valued function on an open interval (a, b). Suppose that there exist points c, d in (a, b) such that $f'(c) < 0$ and $f'(d) > 0$. Prove that there must be some point s in (a, b) such that $f'(s) = 0$. [Hint: again you can not assume that f' is continuous. But suppose that there is no such s: you should be able to use the mean value theorem to show that f is injective, and so obtain a contradiction (see also question 12 from the second question sheet).]

7. Apply the mean value theorem to the function $f(t) = \log(1 + t)$ (choosing suitable endpoints a and b, for example $a = 0$ and $b = x$) to prove that for all $x > 0$,

$$x/(1 + x) < \log(1 + x) < x.$$

8. Determine the following limits if they exist. Show clearly how you obtain your answers.

(a) $\displaystyle\lim_{x\to 0+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ (b) $\displaystyle\lim_{x\to 0} \left(\frac{\cos x}{1 + \sin x}\right)$

(c) $\lim_{x\to +\infty}(\cos(1/x))^{x^2}$

[Hint for (c): substitute $y = 1/x$ and take logs. Use L'Hôpital's rule to find the new limit and then work out what the original limit must have been.]

9. Suppose that the real-valued function f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}\backslash\{0\}$, and that $\lim_{x\to 0} f'(x) = L \in \mathbb{R}$. Prove that f is differentiable at 0 and that $f'(0) = L$. (Hint: use L'Hôpital's rule to investigate $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$, and use the definition of differentiability.)

10. Let f, g be twice-differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(x) = g(x)/x$ for all $x \neq 0$. Given that $g(0) = g'(0) = 0$ and that $g''(0) = 6$, determine $f(0)$ and $f'(0)$.