Exercises for the enthusiast (prize questions)

The following prize questions are only for enthusiasts! Please note that we did not invent these questions. They are well-known and (mostly) quite hard.

As no-one has solved any of the G12RAN enthusiast questions yet this year, those questions will be repeated here, but with increased prizes.

The solutions to these questions need not necessarily have anything to do with real analysis or vector spaces.

For each question, a prize will be awarded for the first correct solution (including justification sufficient to convince Dr Feinstein and Dr Zacharias) by a student registered for either G12RAN Real Analysis or G12VSP Vector Spaces in the academic year 2000-2001. Solutions should be handed directly to Dr Feinstein or Dr Zacharias on or before June 29th 2000.

1. [*16 pounds prize*] Does there exist an uncountable collection of subsets of $\mathbb{N}$ with the property that, for each pair of sets A, B in the collection, either A is contained in B or B is contained in A?

2. [*20 pounds prize*] Let f be a real-valued function defined on the unit square $[0,1]^2$. Suppose that, for each fixed $x \in [0,1]$, $f(x,y)$ is a polynomial in y, and that, for each fixed $y \in [0,1]$, the function $f(x,y)$ is a polynomial in x. Prove that f must be a polynomial in the two variables x and y.

3. [*10 pounds prize*] Show that there DOES exist a function f from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$ such that $f(x,y)$ is NOT a polynomial in x and y, and yet for each fixed $x \in \mathbb{Q}$, $f(x,y)$ is a polynomial in y, and, for each fixed $y \in \mathbb{Q}$, the function $f(x,y)$ is a polynomial in x.

4. [*12 pounds prize*] Show that there is no countably infinite sequence of pairwise disjoint, non-empty closed intervals $[a_n, b_n] \subseteq [0,1]$ whose union is all of $[0,1]$.

5. [*6 pounds prize*] Is there a continuous function f from $[0,1]$ to $\mathbb{R}$ with the following property: for all $x \in [0,1]$, $f(x)$ is rational if and only if x is irrational?

6. [*12 pounds prize*] Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ which is infinitely differentiable, i.e. f can be differentiated as many times as you like at all points of $\mathbb{R}$. Denote the nth derivative of f by $f^{(n)}$ (by convention, $f^{(0)}$ just means f).

   Suppose that for all $a \in \mathbb{R}$ there exists at least one integer (possibly depending on the point a) $n \in \{0,1,2,\ldots\}$ such that $f^{(n)}(a) = 0$ (i.e. at every point of $\mathbb{R}$ there is a derivative of some order of f which is 0 at that point). Prove that f must be a polynomial function.

7. [*16 pounds prize*] Suppose that a rectangle is formed from finitely many smaller rectangles, and that each of the smaller rectangles has the property that at least one of its sides has integer length. Show that the original rectangle must also have at least one side of integer length.

8. [*12 pounds prize*] Let E be a (possibly infinite-dimensional) vector space over $\mathbb{R}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and λ be linear maps from E to $\mathbb{R}$ (such maps are called linear functionals). Prove that the following statements are equivalent (i.e. (i) is true if and only if (ii) is true).

(i) There exist α_1, α_2, ... α_n in $\mathbb{R}$ such that, for all $x \in E$,

$$\lambda(x) = \sum_{k=1}^{n} \alpha_k \lambda_k(x)$$

(i.e λ is a linear combination of λ_1, λ_2, ..., λ_n).

(ii)

$$\bigcap_{k=1}^{n} ker(\lambda_k) \subseteq ker(\lambda).$$