G12VSP Vector Spaces

3. Linear maps

Definition Let F be $\mathbb{R}$ or $\mathbb{C}$ and let U, V be vector spaces over F. A *linear map* (or *linear transformation* or *linear homomorphism*) from U to V is a function T from U to V such the following two conditions are satisfied:
(i) For all $\mathbf{u}_1, \mathbf{u}_2 \in U$ we have $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$;
(ii) For all $\mathbf{u} \in U$ and $\lambda \in F$ we have $T(\lambda \mathbf{u}) = \lambda T(\mathbf{u})$.
There are several equivalent ways to define linear maps. The following proposition is on question sheet 3.

Proposition 3.1 Let F be $\mathbb{R}$ or $\mathbb{C}$, let U, V be vector spaces over F and let $T : U \to V$ be a function. Then the following conditions are equivalent.
(a) T is linear.
(b) For all $\mathbf{x}_1$, $\mathbf{x}_2 \in U$ and λ_1, $\lambda_2 \in F$,

$$T(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) = \lambda_1 T(\mathbf{x}_1) + \lambda_2 T(\mathbf{x}_2).$$

(c) For all $n \in \mathbb{N}$, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in U$ and all $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$,

$$T\left(\sum_{k=1}^{n} \lambda_k \mathbf{x}_k\right) = \sum_{k=1}^{n} \lambda_k T(\mathbf{x}_k).$$

[In other words, a linear map transforms linear combinations of vectors into linear combinations of their images].

If A is an $m \times n$ matrix (m rows and n columns) with entries in F, we shall often write $A = (a_{ij})_{m \times n}$ (or just (a_{ij}) or $[a_{ij}]$) where a_{ij} is the entry in the ith row and jth column of A. Multiplication and addition of matrices (of appropriate sizes) with entries in F are defined as for matrices with real entries. Also, in this module we usually regard elements of F^n as column vectors, so that

$$(x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Given an $m \times n$ matrix $A = (a_{ij})_{m \times n}$ with entries in F, we can define a map T from F^n to F^m by

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

The linear maps met in G1AMAL all have this form, using matrices with real entries. For maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ this includes, for example, the usual geometrical operations of reflection in a line through the origin, or rotation through some fixed angle around the origin (but *not* non-trivial translations. Why not?) These matrix multiplication maps are still linear whether $F = \mathbb{R}$ or $\mathbb{C}$. We shall see (later) that all linear maps from F^n to F^m have this form. If we look at other vector spaces, however, we see that the notion of linear map includes many examples other than matrix multiplication.

Examples
(1) Let P be the usual real vector space of polynomials with real coefficients. Define D from P to P to be the differentiation map, $D(p) = p'$. The fact that D is linear is immediate from the algebra of derivatives, which you met in G12RAN. You can also use differentiation to give you a linear map from V to V where V is the vector space of all infinitely-differentiable real-valued functions on $\mathbb{R}$.
(2) We can also define a linear map d_1 from P to $\mathbb{R}$ by $d_1(p) = p'(0)$. Similarly the map $d_2(p) = p''(0)$ is also linear (*etc.*).

In general, if V is a vector space over F then linear maps from V to the one-dimensional vector space F are called *linear functionals*. The study of linear functionals on infinite-dimensional spaces is important in many advanced areas of Pure Mathematics, especially Functional Analysis.

(3) Another linear functional on P is given by definite integration: we can define ϕ from P to $\mathbb{R}$ by

$$\phi(p) = \int_0^1 p(x)\mathrm{d}x.$$

(You can also use this definite integral to define a linear functional on the vector space of all Riemann-integrable real-valued functions on $[0,1]$).

(4) Another interesting infinite-dimensional vector space is the vector space of all real sequences

$$V = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}\ (n = 1, 2, \ldots)\}$$

with term-wise operations: for $\lambda \in \mathbb{R}$ and $(x_1, x_2, \ldots), (y_1, y_2, \ldots) \in V$ we define

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

and

$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots).$$

Then we can define the *backward shift* (or *left shift*)

$$S((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$$

and the *forward shift* (or *right shift*)

$$R((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, x_3, \ldots).$$

These are easily shown to be linear maps from V to V. One notation used for this space of sequences is $\mathbb{R}^{\mathbb{N}}$, because sequences of real numbers are essentially the same as functions from $\mathbb{N}$ to $\mathbb{R}$.

Definition. Let U and V be sets, and let T be a function from U to V. For $W \subseteq U$ we define the *image of* W *under* T, $T(W)$, by $T(W) = \{T(\mathbf{w}) : \mathbf{w} \in W\}$ (or, equivalently, $T(W) = \{\mathbf{v} \in V :$ there exists $\mathbf{w} \in W$ with $T(\mathbf{w}) = \mathbf{v}\}$.) Recall that a function T from U to V is said to be *injective* (or *1-1*) if whenever $x, y \in U$ and $x \neq y$ then $f(x) \neq f(y)$; T is *surjective* (or *onto*) if $T(U) = V$; T is *bijective* if it is both injective and surjective. (Also, injective functions are called injections, *etc.*). If T is a bijection, then T has an *inverse map* T^{-1} from V to U satisfying $T^{-1}(T(u)) = u$ for all $x \in U$, and $T(T^{-1}(v)) = v$ for all $v \in V$.

Now suppose that U and V are vector spaces over F. To avoid confusion, we shall sometimes denote the zero vectors in U and V by $\mathbf{0}_U$, $\mathbf{0}_V$ respectively. Let T be a linear map from U to V. Then the *image* of T is the set $T(U)$ defined as above (also sometimes denoted by $\mathrm{im}(T)$); the *kernel* (or *null space*) of T, $\ker(T)$ is defined by

$$\ker(T) = \{\mathbf{u} \in U : T(\mathbf{u}) = \mathbf{0}_V\}.$$

It is not hard to show that $\ker(T)$ is always a subspace of U, and $T(U)$ is a subspace of V (see question sheet 3). In fact, whenever W is a subspace of U then $T(W)$ is a subspace of V. One way to see this is to look at *restrictions*.

Definition. Let U, V be vector spaces over F and let T be a linear map from U to V. Let W be a subspace of U. Then the *restriction* of T to W, denoted by $T|_W$ (or $T|W$) is the map from W to V defined by $(T|_W)(\mathbf{w}) = T(\mathbf{w})$ for all $\mathbf{w} \in W$. It is then easy to see that $T|_W$ is linear from W to V, and that the image of $T|_W$ is (of course) $T(W)$.

The following theorem allows us to save some time when checking for injectivity.

Theorem 3.2. Let U, V be vector spaces over F and let T be a linear map from U to V. Then the following statements are equivalent:
(i) T is injective;
(ii) $\ker(T) = \{\mathbf{0}\}$.

In other words, a linear map is 1-1 if and only if its kernel is equal to the trivial subspace.