## G12VSP Vector Spaces

### 3. Linear maps (part II)

**Definition** Let $U$, $V$ be vector spaces over $F$. A map $T$ from $U$ to $V$ is an *isomorphism* (or *linear isomorphism* in case of ambiguity) if it is both linear and also a bijection from $U$ to $V$ (i.e. both injective and onto). In this case we can show that the inverse map $T^{-1}$ is linear from $V$ to $U$.

Indeed $T^{-1} : V \to U$ and if $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ are in $V$, $\lambda \in F$ then there are $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$ such that $T(\mathbf{u}) = \mathbf{v}$, $T(\mathbf{u}_1) = \mathbf{v}_1$ and $T(\mathbf{u}_2) = \mathbf{v}_2$ (namely the respective inverse images). Thus

$$T^{-1}(\mathbf{v}_1 + \mathbf{v}_2) = T^{-1}(T(\mathbf{u}_1) + T(\mathbf{u}_2)) = T^{-1}(T(\mathbf{u}_1 + \mathbf{u}_2)) = \mathbf{u}_1 + \mathbf{u}_2 = T^{-1}(\mathbf{v}_1) + T^{-1}(\mathbf{v}_2)$$

and

$$T^{-1}(\lambda \mathbf{v}) = T^{-1}(\lambda T(\mathbf{u})) = T^{-1}(T(\lambda \mathbf{u})) = \lambda \mathbf{u} = \lambda T^{-1}(\mathbf{v})$$

$T^{-1}$ is in fact an isomorphism from $V$ to $U$. Isomorphisms are also called *invertible* linear maps. If such an isomorphism exists from $U$ to $V$ then we say that $U$ is *isomorphic* to $V$ (sometimes written $U \cong V$). We shall see that there is little difference between isomorphic vector spaces.

Using the *identity transformation* on $V$, $id_V$ (where $id_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$) it is easy to see that every vector space is isomorphic to itself. By the comments above on inverse maps, if $U$ is isomorphic to $V$ then $V$ is also isomorphic to $U$. Also, if $U$ is isomorphic to $V$ and $V$ is isomorphic to $W$ then $U$ is isomorphic to $W$.

We now begin to look at the effects of linear maps on spanning sets and linearly independent sets.

**Proposition 3.3** Let $U$, $V$ be vector spaces over $F$ and let $T$ be a linear map from $U$ to $V$. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in U$ and set $W = \langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \rangle$. Then
(a) $T(W) = \langle T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n) \rangle$, and
(b) if $T$ is injective and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent in $U$ then $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ are linearly independent in $V$. (In this case $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis for $W$ and $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ is a basis for $T(W)$.)

We now come to the important definitions of *rank* and *nullity.*

**Definition** Let $U$, $V$ be vector spaces over $F$ with $U$ finite-dimensional, and let $T$ be a linear map from $U$ to $V$. Then the *rank* of $T$, denoted by $\rho(T)$ is defined by $\rho(T) = \dim(T(U))$ (i.e. $\dim(im(T)))$. The *nullity* of $T$, denoted by $\nu(T)$, is defined by $\nu(T) = \dim(\ker(T))$ (the dimension of the kernel (or null space) of $T$).

**Example** Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 2y \\ -y \end{bmatrix} = y \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$T(\mathbb{R}^2) = \mathbb{R}(1, 2, -1)$ so that $\rho(T) = \dim(T(\mathbb{R}^2)) = 1$ and $\ker(T) = \mathbb{R}(1, 0)$ which means $\nu(T) = \dim(\ker(T)) = 1$. We obtain $\rho(T) + \nu(T) = 1 + 1 = 2 = \dim(\mathbb{R}^2)$ which is the dimension of the domain of $T$. This holds in general!

**Theorem 3.4** (The rank-nullity theorem) Let $U$, $V$, $T$ be as in the preceding definition. Set $n = \dim(U)$. Then

$$\rho(T) + \nu(T) = n,$$

i.e. the dimension of the image of $T$ + the dimension of the kernel of $T$ is equal to the dimension of the domain ($n = \dim(U)$).

We now see the connection between isomorphisms and dimension.

**Proposition 3.5** Let $U$, $V$ be vector spaces over $F$ and suppose that $U$ is isomorphic to $V$. If $U$ is finite-dimensional, then so is $V$ and we have $\dim(V) = \dim(U)$.

**Theorem 3.6** Let $n \in \mathbb{N}$ and let $V$ be an $n$-dimensional vector space over $F$. Then $V$ is isomorphic to $F^n$.

Combining these two results we obtain the following.

**Corollary 3.7** Let $U$, $V$ be finite-dimensional vector spaces over $F$. Then $U$ is isomorphic to $V$ if and only if $\dim(U) = \dim(V)$.

We conclude this chapter by looking at the set of all linear maps from one vector space to another.

**Definition** Let $U$, $V$ be vector spaces over $F$. Then we denote by $L(U, V)$ the set of all linear maps from $U$ to $V$. Linear maps from $V$ to itself (i.e. elements of $L(V, V)$) are called (linear) *endomorphisms* of $V$ (or *linear operators* on $V$).

In fact $L(U, V)$ is itself a vector space: we can add linear maps pointwise, and multiply them by constants. For $S, T \in L(U, V)$ and $\alpha \in F$ we define (for $\mathbf{u} \in U$) $(S + T)(\mathbf{u}) = S(\mathbf{u}) + T(\mathbf{u})$ and $(\alpha S)(\mathbf{u}) = \alpha(S(\mathbf{u}))$. You might want to convince yourselves that the eight vector space axioms are all satisfied!

We can also define composite linear maps: given vector spaces $U$, $V$, $W$ over $F$ and linear maps $T$ from $U$ to $V$ and $S$ from $V$ to $W$, we can define the composite map $S \circ T$, also written simply as $ST$, by $(ST)(\mathbf{u}) = (S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$. It is easy to check that $ST$ is then a linear map from $U$ to $W$.

In the case of linear endomorphisms (operators) on $V$ this allows us to take powers and even polynomials of endomorphisms: for $T \in L(V, V)$, we denote $TT (= T \circ T)$ by $T^2$, and similarly $T^3 = TTT$ *etc.*. Regarding $T^0$ as the identity transformation on $V$, $id_V$, we can then apply polynomials to $T$. If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ then we define $p(T)$ by $p(T) = a_0 id_V + a_1 T + \cdots + a_n T^n$ (note that $p(T)$ is then also an endomorphism of $V$).