G13MIN: Measure and Integration
Blow-by-blow account of the module as taught in 2005-6

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Other staff involved: None

Lecture 1: **Introductory session.** Sizes of sets. Finite sets, countable sets, uncountable sets mentioned briefly. Total length of finite disjoint unions of intervals and unions of sequences of pairwise disjoint intervals. Examples of open subsets of $\mathbb{R}$ which contain $\mathbb{Q}$: such open sets must have strictly positive total length, but their total length can be arbitrarily small (using a union of a suitable sequence of open intervals). In contrast to this, although this fact is not obvious (it will follow from our later theory), any open subset of $\mathbb{R}$ which contains $[0,1] \setminus \mathbb{Q}$ (the set of irrational numbers in $[0,1]$) must have total length at least 1. Some pathological subsets of $\mathbb{R}$ can not be assigned a length in a satisfactory way. (There are similar problems for area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$, etc.) Brief discussion of the Banach-Tarski paradox.

Lecture 2: **Functions and sets.** Domain and codomain. Injective (one-one) surjective (onto) and bijective functions (injections, surjections, bijections). Examples and non-examples. Inverse functions. Finite sets and infinite sets. A bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Lecture 3: **Revision session on mathematical analysis.** Questions and answers, including a discussion of pointwise and uniform convergence for sequences of functions.

Lecture 4: Two sets have the same cardinality (or the same power) if there is a bijection between them. Countability and uncountability. The connection between sequences in X and functions from $\mathbb{N}$ to X. Countability in terms of sequences. $\mathbb{Z}$ is countable. Which real numbers have two different decimal expansions? Uncountability of $[0,1)$ (Cantor diagonalization argument). The uncountability of $\mathbb{R}$ is similar, or may be deduced from this. Many standard results on countability may be found on the first question sheet.

Lecture 5: Further standard results about countability discussed: finite unions, countable unions and products of countable sets are countable (details on question sheet). Failure of proof by induction to help in certain situations. Brief discussion of harder results including the Schroeder-Bernstein Theorem: see books if interested.
The extended real line. This is $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, a totally ordered set. Intervals, upper bounds, lower bounds. Most details left as exercises. EVERY subset E of $\overline{\mathbb{R}}$ has an infimum and a supremum in $\overline{\mathbb{R}}$, denoted by $\inf(E)$ and $\sup(E)$ respectively. The minus operator $x \mapsto -x$. Sequences in $\overline{\mathbb{R}}$: the limit infimum and limit supremum of a sequence ($\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$). Warning over notation: do not let the lim get separated from the inf/sup! For example, $\lim_{n\to\infty}(\sup x_n)$ does not make sense. Examples and standard properties.

Lecture 6: Convergent sequences in $\overline{\mathbb{R}}$ defined in terms of $\liminf$ and $\limsup$: this extends the usual definition of convergence in $\mathbb{R}$, but sequences which previously diverged to $+\infty$ now **converge** to $+\infty$ in $\overline{\mathbb{R}}$. Equivalent definitions of convergence to $\pm\infty$ in $\overline{\mathbb{R}}$. A standard homeomorphism between $\mathbb{R}$ and $(-1,1)$ extends to a homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$. The monotone sequence theorem in $\overline{\mathbb{R}}$. The algebra of limits in $[0,\infty]$ (with limitations: see Question Sheet 1). Arithmetic in $\overline{\mathbb{R}}$: addition and subtraction (where possible) and multiplication in $\overline{\mathbb{R}}$. Series in $\overline{\mathbb{R}}$. Series with terms in $[0,\infty]$. Fact: series with non-negative terms can be rearranged arbitrarily and still give the same sum (finite or infinite). (Some special cases are proved in the printed notes. These results also follow from results on integration in Chapter 4.)

Lecture 7: Open sets in $\mathbb{R}$ are countable unions of open intervals. Properties and examples of open sets. Closed sets in $\mathbb{R}$. Examples. Properties. Open subsets of $\overline{\mathbb{R}}$ and closed subsets of $\overline{\mathbb{R}}$.
Classes of sets. Motivation recalled: we would like to measure the area of all subsets of $\mathbb{R}^2$, but this can not be done in a satisfactory way. There are similar problems for 'total length' for subsets of $\mathbb{R}$, as we will see shortly. We will need to work, instead, with a large class of subsets of $\mathbb{R}$ (or $\mathbb{R}^2$) which includes essentially all 'sensible' sets. In particular, we will have no problems with open sets and with closed sets. Symmetric difference introduced, various equivalent definitions including addition of characteristic functions modulo 2. The power set of X, $\mathcal{P}(X)$ (or 2^X). Levels of abstraction, notation e.g $1 \in \mathbb{N}$, $\{1,2\} \subseteq \mathbb{N}$, $\{1,2\} \in \mathcal{P}(\mathbb{N})$, $\{\{1,2\},\{3,4\}\} \subseteq \mathcal{P}(N)$, $\{\{1,2\},\{3,4\}\} \in \mathcal{P}(\mathcal{P}(\mathbb{N}))$, etc. Beware of ambiguity in: 'contains' and 'contained in'. Safer to use 'element of' and 'subset of', as appropriate.

Lecture 8: Semi-rings of sets: the set of all intervals in $\mathbb{R}$; the semiring $P = \{(a,b] : a,b \in \mathbb{R}, a \leq b\}$ of half-open intervals (open at the left); similarly, half-open rectangles in $\mathbb{R}^2$, etc. Rings of sets. Elementary figures in $\mathbb{R}$: finite (disjoint) unions of half-open intervals from P. Elementary figures in $\mathbb{R}^2$ and in $\mathbb{R}^n$. The ring generated by a semi-ring. Fields (or algebras) of subsets of a set X (also called 'fields on X'). Alternative definitions, deductions from axioms. Definition and examples of σ-fields of subsets of a set X.

Lecture 9: Revision of semi-rings, rings, fields and σ-fields of subsets of a set X. Indexing sets and intersections of indexed families of collections of subsets of X. Whenever you have some σ-fields of subsets of X, say $\mathcal{F}_\gamma$ ($\gamma \in \Gamma$), then $\bigcap_{\gamma \in \Gamma} \mathcal{F}_\gamma$ is also a σ-field of subsets of X. The σ-field on X generated by a collection $\mathcal{C}$ of subsets of X, denoted in this module by $\mathcal{F}(\mathcal{C})$ or, to avoid ambiguity, $\mathcal{F}_X(\mathcal{C})$. This is the smallest possible σ-field on X which includes all of the sets in $\mathcal{C}$. More formally, $\mathcal{F}_X(\mathcal{C})$ is a σ-field on X, $\mathcal{C} \subseteq \mathcal{F}_X(\mathcal{C})$ and, whenever $\mathcal{G}$ is a σ-field on X such that $\mathcal{C} \subseteq \mathcal{G}$ then we have also $\mathcal{F}_X(\mathcal{C}) \subseteq \mathcal{G}$. Proof of the existence and properties of $\mathcal{F}_X(\mathcal{C})$. Comparison of $\mathcal{F}_X(\mathcal{C})$ and $\mathcal{F}_Y(\mathcal{C})$ for a collection $\mathcal{C}$ of subsets of X when $X \subseteq Y$ (exercise: see also Question Sheet 2). What can you say about a σ-field on $\mathbb{R}$ if it includes all of the open subsets of $\mathbb{R}$? The σ-field, $\mathcal{B}$, of all Borel sets in $\mathbb{R}$ (also called Borel subsets of $\mathbb{R}$ or Borel measurable subsets of $\mathbb{R}$): $\mathcal{B}$ is the σ-field generated by the collection of all open subsets of $\mathbb{R}$. Borel subsets of $\overline{\mathbb{R}}$ (and other metric spaces). Examples of Borel subsets of $\mathbb{R}$, including open sets, closed sets, $\mathbb{Q}$, Brief comments on transfinite induction (beyond the scope of this module, but see books if interested). Introduction to the Cantor set.

Lecture 10: More details on the Cantor set. The Cantor function. Countable intersections of open sets (G_δ sets) and countable unions of closed sets (F_σ sets) are Borel sets. There are many other Borel sets. Proof that $\mathcal{F}_{\mathbb{R}}(P) = \mathcal{B}$ (with P and $\mathcal{B}$ as above). Related facts are on Question Sheet 2. All these proofs are based on the fact that whenever $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$ and a σ-field $\mathcal{G}$ on $\mathbb{R}$ is such that $\mathcal{C} \subseteq \mathcal{G}$ then the σ-field on $\mathbb{R}$ generated by $\mathcal{C}$, $\mathcal{F}_{\mathbb{R}}(\mathcal{C})$, must also be $\subseteq \mathcal{G}$. Short cuts for proving $\mathcal{F}(\mathcal{C}_1) = \mathcal{F}(\mathcal{C}_2)$: $\mathcal{F}(\mathcal{C}_1) \subseteq \mathcal{F}(\mathcal{C}_2)$ if and only if $\mathcal{C}_1 \subseteq \mathcal{F}(\mathcal{C}_2)$. Definition of Lebesgue outer measure λ^* (defined for all subsets of $\mathbb{R}$) and Lebesgue measure λ on the Borel sets (obtained by restricting λ^* to the Borel sets). Properties stated (see Section 5 for details). Brief revision of equivalence relations. A non-Borel set described briefly. Details next time.

Lecture 11: Revision of λ^* and its properties. Translation of sets: every translate of a Borel set is a Borel set. Brief discussion of continuous pre-images and images. Brief mention of the Lebesgue measurable sets (see Section 5). A 'non-measurable' subset of $\mathbb{R}$: its properties, and associated problems for the measurement of 'total length'. Similar problems with area and volume. Motivation for measures on σ-fields: the best we can hope for is a measure of size that behaves well on a large class of sets.

Lecture 12: Definition of (positive) measure on a collection $\mathcal{C}$ of subsets of X (with $\emptyset \in \mathcal{C}$). Examples: counting measure (on any set), Lebesgue measure λ on the Borel sets $\mathcal{B}$ (the restriction of Lebesgue outer measure λ^* to $\mathcal{B}$). Properties of Lebesgue outer measure and Lebesgue measure will be assumed in Sections 1-4, but will be justified in Section 5. Measurable spaces and measure spaces. Properties of measures on rings: countable additivity (part of definition), finite additivity, Monotonicity and countable subadditivity. Finite measures, probability measures, σ-finite measures.

Lecture 13: More examples of measures: point-mass measures and the biggest possible measure on $\mathcal{P}(X)$. Length on the semi-ring P discussed earlier: $\mu((a,b]) = b - a$ (see Section 5 for the proof that this is a measure). This does NOT work if you work with 'intervals in $\mathbb{Q}$', $(a,b] \cap \mathbb{Q}$. Taking the length of $(a,b] \cap \mathbb{Q}$ to be $b - a$ leads to a measure which is finitely additive but not countably additive (see question sheets). One main difference here is that $[0,1]$ is (sequentially) compact, but $[0,1] \cap \mathbb{Q}$ is not. Sums and multiples of measures. Various types of measures: positive measures (the measures used in this module), complex measures, real measures, signed measures. Hahn decomposition for complex/signed measures stated in the form $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ (where μ_i are positive measures, $1 \leq i \leq 4$). Continuity properties of measures on rings discussed, with emphasis on the case of nested increasing unions of sets.

Lecture 14: Continuity properties of measures on rings: measures (when defined) of countable unions and countable intersections of sets in rings. Properties that hold almost everywhere e.g. almost every real number is irrational (because $\lambda(\mathbb{Q}) = 0$, where λ is Lebesgue measure on $\mathbb{R}$). Exceptional sets/bad sets of measure 0 will not matter in our theory: the integral of a function is unaffected by the values it takes on such a set (see later). We can combine countably many exceptional sets of measure 0 to form one 'big' exceptional set which still has measure 0. Equivalence (almost everywhere equality) of functions on measure spaces. What this means for some specific measures. Revision of Riemann integration: approximation of functions from below and above using step functions.

Lecture 15: The idea behind the Lebesgue integral: start with finite linear combinations of characteristic functions of sets more general than intervals (can use any measurable sets). These will be easy to integrate (in particular we will have no problem integrating $\chi_{\mathbb{Q}}$, which gave problems with the Riemann integral). Simple functions on X: definition (n.b. simple functions are real-valued), examples, standard form (using the distinct values, partition the set X and so form a finite linear combination of characteristic functions). Sums, products and linear combinations of simple functions are still simple functions. Every finite linear combination of characteristic functions is a simple function (even if the sets do not form a partition of X or the coefficients are not distinct). Measurable spaces, measurable sets and measurable functions. Continuous functions, images and pre-images revised. Topological definition of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (in terms of pre-images of open sets). Measurable functions from one measurable space to another. Using the Borel sets on the codomain it is enough to check the pre-images of open sets. Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is (Borel) measurable. (By default we use the Borel sets as our σ-field on $\mathbb{R}$.) Characteristic functions of measurable sets are measurable functions, while those of non-measurable sets are non-measurable functions. Most sensible functions are measurable, but the characteristic function of the non-Borel set we constructed earlier is a non-measurable function on $\mathbb{R}$.

Lecture 16: Measurability of functions taking values in $\overline{\mathbb{R}}$. Four conditions equivalent to measurability (from X to $\mathbb{R}$ or $\overline{\mathbb{R}}$), including that, for all $a \in \mathbb{R}$, the set $\{x \in X : f(x) \leq a\}$ be a measurable set (i.e. a set that is in the σ-field we are using on X). The function $-f$ is measurable if and only if f is measurable. The pointwise $\sup$, $\inf$, $\limsup$ and $\liminf$ of any sequence of measurable functions is measurable. Hence every pointwise limit of a sequence of measurable functions is a measurable function.

Lecture 17: When defined, a sum of two measurable functions is measurable. Simple measurable functions (measurable simple functions): finite linear combinations of characteristic functions of measurable sets. A sum or product of two simple measurable functions is again a simple measurable function. Sketch shown of approximation of the function $f(x) = x$ by simple functions on $[0, \infty)$. Monotone approximation from below of non-negative measurable functions using non-negative simple measurable functions. Deduction (from the corresponding result for simple measurable functions) of the fact that the product of two non-negative measurable functions is measurable. Many results for general measurable functions can be deduced in the same way using the results for simple measurable functions and this method of approximation.

Lecture 18: The pointwise maximum of two $\overline{\mathbb{R}}$-valued measurable functions is a measurable function. Decomposition of $\overline{\mathbb{R}}$-valued functions into positive and negative parts: $f = f^+ - f^-$. The function f is measurable if and only if both f^+ and f^- are. Definition of the (Lebesgue) integral of non-negative, simple measurable functions (notation: $I_E(s, \mu)$ [non-standard]). Connection with Riemann integrals of step functions. Brief discussion of some standard facts (mostly intuitively obvious, proofs in printed notes or on Question Sheet 4, some proofs discussed in lectures).

Lecture 19: When s is a non-negative, simple measurable, then the function $\phi(E) = I_E(s, \mu)$ is a measure on $\mathcal{F}$. Definition of the Lebesgue integral of a non-negative measurable function, $\int_E f \, \mathrm{d}\mu$. In particular this gives the same value as before for simple measurable functions: $I_E(s, \mu) = \int_E s \, \mathrm{d}\mu$. Thus we may safely switch to the new notation, but maintain our old results. For example, when s is non-negative, simple measurable, then the function $\psi(E) = \int_E s \, \mathrm{d}\mu$ is a measure on $\mathcal{F}$. Brief discussion of standard facts about integrals of non-negative, measurable functions (proofs in printed notes, most follow directly from the definitions and the results for non-negative, simple measurable functions).

Lecture 20: Further elementary properties of the integral. Revision of continuity properties of measures. Statement and proof of the Monotone Convergence Theorem.

Lecture 21: Typical application of MCT: deduction of less elementary facts about integrals of non-negative, measurable functions using facts about simple measurable functions and monotone approximation. Integral of a sum of two non-negative, measurable functions:

$$\int_X (f + g) \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

For a non-negative measurable function f and $\alpha \in [0, \infty)$ we have

$$\int_X \alpha f \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu$$

(this may also be proved by elementary means). For non-negative measurable functions f_n,

$$\int_X \left(\sum_{n=1}^{\infty} f_n\right) \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, \mathrm{d}\mu\right).$$

For any non-negative measurable function f, the function $\Phi(E) = \displaystyle\int_E f \, \mathrm{d}\mu$ is a measure on $\mathcal{F}$. Counting measure on $\mathbb{N}$: connection between integrals and series. In particular, another proof of the fact that for non-negative extended real numbers $a_{n,k}$,

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{n,k}\right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{n,k}\right).$$

Lecture 22: Statement and proof of Fatou's Lemma. Defined (where possible) the integral over E of a measurable $\overline{\mathbb{R}}$-valued function f to be $\displaystyle\int_E f \mathrm{d}\mu = \int_E f^+ \mathrm{d}\mu - \int_E f^- \mathrm{d}\mu$. The (measurable) function f is integrable on E if both f^+ and f^- have finite integral on E. In particular, a function $f : X \to \overline{\mathbb{R}}$ is said to be integrable if it is measurable and integrable on X. For a measurable function f, f is integrable if and only if $|f|$ is integrable. Connection with absolutely convergent series of real numbers. The set of integrable functions f such that f take values in $\mathbb{R}$ (non-standard) is denoted by $L^1(\mu)$ (or $L^1(X, \mu)$ or $L^1(X, \mathrm{d}\mu)$). Most authors allow f to be $\overline{\mathbb{R}}$-valued, but this makes no real difference to the theory. For integrable functions

$$\left|\int_X f \, \mathrm{d}\mu\right| \leq \int_X |f| \, \mathrm{d}\mu.$$

For f in $L^1(\mu)$ and $\alpha \in \mathbb{R}$, we have

$$\int_X (\alpha f) \, \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu.$$

Warning that, in general, $(f+g)^+ \neq f^+ + g^+$.

Lecture 23: $L^1(\mu)$ is a vector space of functions on X, and, for f, g in $L^1(\mu)$,

$$\int_X (f+g) \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

Statement and proof of the Dominated Convergence Theorem (DCT). From question sheets: sets of measure zero have no effect on integration. Countable unions of sets of measure 0 still have measure 0. As a result, conditions in convergence theorems are only required to hold almost everywhere.

Lecture 24: **Problem class/Tutorial session** on the main three theorems of Chapter 4: the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem (discussed in the context of Riemann integrals of functions and sums of series). Students worked in groups to find counterexamples when conditions of the theorems are weakened, and an example where the inequality in Fatou's Lemma is strict. Answers were discussed.

Lecture 25: The connection between the Riemann integral and the Lebesgue integral w.r.t. Lebesgue measure λ (see below). The two agree for all Riemann integrable functions (proof based on approximation of a Riemann integrable function from above and below by step functions: see printed notes for more details). This allows us to use the notation $\int_a^b f(x)\ \mathrm{d}x$ for the Lebesgue integral $\int_{[a,b]} f\ \mathrm{d}\lambda$ of a Lebesgue integrable function (even if it is not Riemann integrable).
Outer measures and the construction of Lebesgue measure Definition and examples of outer measures. Definition of μ^*-measurable sets for an outer measure μ^*. Statements of some standard results (see printed notes for full details): the set of μ^*-measurable sets is a σ-field and the restriction of μ^* to this σ-field is a complete measure. Lebesgue outer measure λ^* is an outer measure on $\mathbb{R}$ and the half-open intervals $(a, b]$ are λ^*-measurable with $\lambda^*((a, b]) = b - a$. The Lebesgue measurable sets (λ^*-measurable sets) are thus a σ-field $\mathcal{F}$ which includes all the Borel sets. Lebesgue measure λ on $\mathbb{R}$ is the restriction of λ^* to $\mathcal{F}$. The non-Borel set we constructed earlier is, in fact, not Lebesgue measurable either.

Lecture 26: Elementary results concerning finite unions of intervals and their lengths. Proof, using Bolzano-Weierstrass, of a compactness result for closed intervals $[a, b]$: every cover using a sequence of open intervals has a finite sub-cover. Countable unions of intervals: length gives a measure on our favourite semi-ring P of half-open intervals. For the equivalent result concerning area in $\mathbb{R}^2$, see Question Sheet 5. (The same can be done in $\mathbb{R}^n$.). Discussion of extensions of measures from semi-rings to rings (full details in printed notes). In particular, we may extend our measure on μ to obtain a measure on the ring $\mathcal{E}$ generated by P. Recalled outer measures (as introduced in previous lecture). Outer measures will enable us to extend measures from semi-rings to the σ-field they generate (and beyond), and this will give us Lebesgue measure when we start with length on P.

Lecture 27: Revision of concept of measurability with respect to an outer measure μ^*. Lemma: The collection of μ^*-measurable sets is a field on which μ^* is finitely additive. Theorem stated (see printed notes for details): The collection of μ^*-measurable sets is a σ-field on which μ^* is countably additive. The restriction of μ^* to this σ-field is thus a measure. Moreover, this measure is complete. How to use a measure μ on a semi-ring or a ring to define an outer measure μ^*. (We will use length on intervals/elementary sets to obtain Lebesgue outer measure on $\mathbb{R}$.) Statement of the Extension Theorem: every measure on a ring can be extended to a measure on a σ-field containing the ring.

Lecture 28: Proof of Extension Theorem using outer measures. The measure obtained is complete. Combining our results gives another version of the Extension Theorem: every measure on a semi-ring may be extended to a complete measure on a σ-field containing the semi-ring, using the standard outer measure construction. The particular case of Lebesgue outer measure and Lebesgue measure. A reminder of some of the useful properties of Lebesgue measure. Student Evaluation of Teaching forms.

Lecture 29: Problem Class on measures and outer measures + Question and Answer session.