G13MIN: MEASURE AND INTEGRATION

LECTURE NOTES

Section 1: The Extended Real Line

Notation:

The extended real line $\overline{\mathbb{R}} \equiv [-\infty, \infty]$ is defined as follows. We start with $\mathbb{R}$ and adjoin two new points, denoted by ∞ and $-\infty$. Then $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We define a relation $\leqslant$ on $\overline{\mathbb{R}}$ by

$$\begin{aligned} -\infty &\leqslant x \quad \text{all } x \\ x &\leqslant \infty \quad \text{all } x \end{aligned}$$

and for $x, y \in \mathbb{R}$, $x \leqslant y$ has the usual meaning.

This gives us a total order on $\overline{\mathbb{R}}$ extending the usual order on $\mathbb{R}$. We can now immediately extend our usual notation of intervals: e.g. $\mathbb{R}=(-\infty,\infty)$ and $\overline{\mathbb{R}} = [-\infty,\infty]$, etc.

Definition 1.1. Let $E \subseteq \overline{\mathbb{R}}$. Then E always has a least upper bound in $\overline{\mathbb{R}}$, the *supremum* of E denoted by sup (E). Also E has a greatest lower bound, the *infimum* inf (E). We have $\sup(E) \geqslant \inf(E)$, except when $E = \varnothing$ (note that $\sup(\varnothing) = -\infty$, $\inf(\varnothing) = \infty$).

For $x \in \overline{\mathbb{R}}$ we define $-x$ to have its usual meaning if $x \in \mathbb{R}$. We define

$$-x = \begin{cases} -\infty & \text{if } x = \infty \\ \infty & \text{if } x = -\infty \end{cases}$$

$x \mapsto -x$ is a bijection from $\overline{\mathbb{R}}$ to itself. Now let (x_n) be a sequence of elements of $\overline{\mathbb{R}}$. For each $n \in \mathbb{N}$ we set

$$S_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

$$s_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

We have $s_n \leqslant S_n$ for all n and

$$S_1 \geqslant S_2 \geqslant S_3 \geqslant \ldots \quad \text{and} \quad s_1 \leqslant s_2 \leqslant s_3 \leqslant \ldots .$$

We define

$$\lim_{n\to\infty} \sup(x_n) = \inf\{S_n \colon n \in \mathbb{N}\}$$

$$\lim_{n\to\infty} \inf(x_n) = \sup\{s_n \colon n \in \mathbb{N}\}.$$

We then have

$$\lim_{n\to\infty} \inf(x_n) \leqslant \lim_{n\to\infty} \sup(x_n).$$

Example 1.2. Let $a \in \mathbb{R}$. Consider the sequence $x_n = a^n$. For $-1 < a < 1$, x_n will converge to 0 as

$n \to \infty$: this gives $\liminf_{n\to\infty}(x_n) = \limsup_{n\to\infty}(x_n) = 0$.

If $a = -1$, the sequence x_n is

$$-1, 1, -1, 1, \ldots$$

thus $\liminf_{n\to\infty}(x_n) = -1$, $\limsup_{n\to\infty}(x_n) = 1$.

Suppose $a < -1$. Then

$$\liminf_{n\to\infty}(a^n) = -\infty,$$

$$\limsup_{n\to\infty}(a^n) = \infty.$$

The remaining two cases are left as an exercise for the reader.

Proposition 1.3. Let (x_n) be a sequence of elements of $\overline{\mathbb{R}}$. Then

$$\liminf_{n\to\infty}(x_n) = -\limsup_{n\to\infty}(-x_n).$$

For the proof of this, see the Question Sheet 1.

Definition 1.4. We say that a sequence (x_n) *converges* to x in $\overline{\mathbb{R}}$ if

$$\liminf_{n\to\infty}(x_n) = \limsup_{n\to\infty}(x_n) = x.$$

It is standard that $(-1, 1)$ is *homeomorphic* to $\mathbb{R}$: there exists a bijective map f from $(-1,1)$ to $\mathbb{R}$ so that both f and f^{-1} are continuous (such a map f is called a *homeomorphism*). For example, we can use the map

$$f(x) = \frac{x}{1 - |x|}.$$

The inverse map (from $\mathbb{R}$ to $(-1, 1)$ is the map

$$y \mapsto \frac{y}{1 + |y|}.$$

If we set $f(-1) = -\infty$, $f(1) = \infty$ then we have extended f to a bijection from $[-1, 1]$ onto $[-\infty, \infty]$. Also f is strictly increasing:

$$x < y \Rightarrow f(x) < f(y).$$

Using the function f, we can define, for $x, y \in [-\infty, \infty]$

$$d(x, y) = |f^{-1}(x) - f^{-1}(y)|$$

For those who attended G13MTS Metric and Topological Spaces: $(\overline{\mathbb{R}}, d)$ is then a metric space isometric to $[-1, 1]$. Convergence with respect to this metric agrees with the above definition of convergence in $\overline{\mathbb{R}}$. We will not use this metric often, but we will occasionally talk about *open* subsets of $\overline{\mathbb{R}}$ (see below).

Proposition 1.5. Let $(x_n) \subseteq \overline{\mathbb{R}}$, and suppose that $x_n \leqslant x_{n+1}$ for all n. Then x_n converges in $\overline{\mathbb{R}}$.

Proof. This is essentially a restatement of the standard result for non-decreasing sequences of real numbers.

Arithmetic in $\overline{\mathbb{R}}$

For $x, y \in \mathbb{R}$ we already have defined $x \cdot y$ and $x+y$. For $x \in [0,\infty]$ we *define*

$$x+\infty = \infty+x = \infty$$

$$x\cdot\infty = \infty . x = \begin{cases} \infty & (x > 0), \\ 0 & (x = 0). \end{cases}$$

With these operations on $[0,\infty]$ the associative and distributive laws are satisfied.

$$\left.\begin{aligned} (x\cdot y)\cdot z &= x\cdot(y\cdot z) \\ (x+y)+z &= x+(y+z) \\ x\cdot(y+z) &= x\cdot y+x\cdot z \end{aligned}\right\} \quad x,y,z \in [0,\infty].$$

(The reader should check this.)

On the whole of $\overline{\mathbb{R}}$ addition is a problem. We cannot define $-\infty+\infty$ satisfactorily.

We *can* define

$$\begin{aligned} x+\ \infty &= \ \infty+x = \ \infty \quad (x \neq -\infty) \\ x+-\infty &= -\infty+x = -\infty \quad (x \neq \infty) \end{aligned}$$

$$0\cdot x = x\cdot 0 = 0 \quad \text{for all } x$$

$$(-\infty)\cdot x = \begin{cases} -\infty & x > 0 \\ 0 & x = 0 \\ \infty & x < 0 \end{cases}$$

$$\infty\cdot x = \begin{cases} \infty & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$$

We denote $x+(-y)$ by $x-y$ for short, when it is defined.

Proposition 1.6. Let $(x_n), (y_n)$ be sequences in $[0,\infty]$. Then

(i) if $x_n \to x$ and $y_n \to y$, then $x_n+y_n \to x+y$,

(ii) if $x_n \uparrow x$ and $y_n \uparrow y$ then $x_n \cdot y_n \to xy$.

Proof. (i) is easy. For (ii) see the first question sheet. □

We can now consider series

$$\sum_{n=1}^{\infty} x_n,$$

where $x_n \in \overline{\mathbb{R}}$.

Assuming that $-\infty, \infty$ do not both occur, then, for each k, we have a partial sum

$$\sum_{n=1}^{k} x_n .$$

Definition 1.7. If the sequence of partial sums converges in $\overline{\mathbb{R}}$ as $k \to \infty$, then we say the series *converges* in $\overline{\mathbb{R}}$.

Proposition 1.8. If $x_n \in [0,\infty]$ all n, then by Proposition 1.5 the series $\sum_{n=1}^{\infty} x_n$ must converge to an element of $[0,\infty]$. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges in $\mathbb{R}$, is convergent in $\overline{\mathbb{R}}$. Of course, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Proposition 1.9. Let $a_{nk} \in [0,\infty]$ $(n,k \in \mathbb{N})$. Then

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{nk} .$$

Proof. By symmetry it is enough to show the left hand side is $\leqslant$ right hand side. Using Proposition 1.6(i) and induction on m we see easily that

$$\begin{aligned} \sum_{n=1}^{m} \sum_{k=1}^{\infty} a_{nk} &= \sum_{k=1}^{\infty} \sum_{n=1}^{m} a_{nk} \qquad \text{for all } m \in \mathbb{N} \\ &\leqslant \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{nk} . \end{aligned}$$

Letting $m \to \infty$ we obtain

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} \leqslant \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{nk} \qquad \text{as required.}$$

□

Proposition 1.10. (Used later to extend measures.) Given a_{ik}, $i = 1,2,\ldots$, $1 \leqslant k \leqslant n_i$, enumerate the pairs $\{(i,k)\colon 1 \leqslant k \leqslant n_i,\ i \in \mathbb{N}\}$ as (i_t, k_t) $t = 1,2,\ldots$. Then

$$\sum_{i=1}^{\infty} \sum_{k=1}^{n_i} a_{ik} = \sum_{t=1}^{\infty} a_{i_t k_t} .$$

Proof. Each side is a limit of finite sums, each of which is less than or equal to the infinite sum on the other side. □

Definition 1.11 A subset U of $\mathbb{R}$ is said to be *open in* $\mathbb{R}$ if it can be written as a countable union of (possibly empty) open intervals (a_n, b_n) where a_n and b_n are real numbers with $a_n \leqslant b_n$. (Note that the empty set is open, as is $\mathbb{R}$ itself, and so are unbounded open intervals such as $(0,\infty)$ etc.) A subset E of $\mathbb{R}$ is said to be *closed in* $\mathbb{R}$ if $\mathbb{R} \setminus E$ is open in $\mathbb{R}$.

The following properties of open sets/closed sets are standard: if you are not familiar with them you can look in books or prove these facts as exercises. Every countable union of open sets is open (in fact so are uncountable unions). Every countable intersection of closed sets is closed. Finite intersections of open sets are open, and finite unions of closed sets are closed. Countable intersections of open sets need not be open.

Occasionally we may want to discuss open sets or closed sets in the extended real line instead. This can be defined using the metric above, or as follows.

Definition 1.12 A subset U of $\overline{\mathbb{R}}$ is said to be *open in* $\overline{\mathbb{R}}$ if it is a finite union of sets taken from any of the following three collections of subsets of $\overline{\mathbb{R}}$: $\{V \subseteq \mathbb{R} : V \text{ is open in } \mathbb{R}\}$, $\{[-\infty, a) : a \in \mathbb{R}\}$, $\{(b, \infty] : b \in \mathbb{R}\}$.

Note that in fact you never need more than three of these sets: a typical open subset of $\overline{\mathbb{R}}$ including the points $-\infty$ and ∞ has the form $[-\infty, a) \cup U \cup (b, \infty]$ where a and b are real numbers, and U is an open subset of $\mathbb{R}$.

A subset E of $\overline{\mathbb{R}}$ is *closed in* $\overline{\mathbb{R}}$ if $\overline{\mathbb{R}} \setminus E$ is open in $\overline{\mathbb{R}}$.